New vortex types and soliton substructures

PhD thesis

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Abstract

This thesis is concerned with topological solitons of static type in supersymmetric gauge theories and especially vortices of Abelian and non-Abelian kind with an arbitrary gauge symmetry. We put emphasis on the gauge theories with orthogonal and unitary-symplectic gauge symmetry but in many cases we keep the group arbitrary. We investigate the moduli spaces of vortices of local and semi-local kind where the latter are related to lumps in the low-energy effective theories, that in turn are related to the Kähler quotient construction, which we obtain explicitly. The moduli space is studied systematically using the so-called moduli matrix formalism which exhausts all moduli of the vortex and we cross-check with an index theorem calculation. We consider theories both with and without Chern-Simons interactions for the gauge fields, where in the case of only the Yang-Mills kinetic term and of only the Chern-Simons term present in the theory, we find the same moduli spaces of vortices for a class of theories. Finally, we investigate so-called fractional vortices which are energy-substructures appearing in various theories due to either deformations or singularities on the target space of the soliton's low-energy effective theories.

Riassunto

Questa tesi tratta di solitoni topologici statici nelle teorie di gauge supersimmetriche, e in particolare dei vortici Abeliani e non-Abeliani con simmetria di gauge arbitraria. Ci occupiamo specialmente delle teorie di gauge con simmetria ortogonale e simplettica unitaria, lasciando però il gruppo più generico possibile nella maggior parte dei casi. Investighiamo sopratutto lo spazio dei moduli dei vortici locali e non locali, dove questi ultimi sono connessi a dei lumps nelle teorie efficaci di bassa energia, che sono a loro volta connessi ai quoziente di Kähler che abbiamo trovato esplicitamente. Lo spazio dei moduli è stato studiato sistematicamente usando il cosiddetto formalismo della matrice dei moduli e facendo un controllo incrociato con il risultato ottenuto utilizzando il teorema dell'indice. Consideriamo poi le teorie con e senza il termine di interazione di Chern-Simons per i campi di gauge, e troviamo che nel caso della teoria con solo il termine di Chern-Simons e il caso con solo il termine di Yang-Mills, hanno lo stesso spazio dei moduli. Infine, investighiamo i cosiddetti vortici frazionari, sottostrutturati nella densità d'energia, che appaiano in varie teorie a causa o delle deformazioni o delle singolarità sullo spazio target della teoria efficace del solitone.

Resumé

Denne thesis drejer sig om topologiske, statiske solitoner i supersymmetriske gauge teorier, og specielt Abelske og ikke-Abelske vortex-løsninger med arbitrær gauge symmetri. Vi lægger vægt på gaugeteorier med ortogonal og unitær-symplektisk gauge symmetri, men i mange tilfælde lader vi gruppen være fuldstændig arbitrær. Vi undersøger modulimangfoldigheden af lokale og semi-lokale vortex-løsninger, hvor de sidstnævnte har forbindelser til klumpe-løsninger i effektive lav-energi-teorier, som er forbundet med Kähler-kvotient-konstruktioner som vi har fundet eksplicit. Modulimangfoldigheden er blevet studeret systematisk ved hjælp af den såkaldte moduli matrix formalisme, som finder alle moduli for vortexen og vi laver ydermere et krydstjek med en indeksteoremsberegning. Vi studerer både teorier med og uden Chern-Simons vekselvirkninger for gauge felterne, og vi finder ud af at i tilfældet med kun et Yang-Mills-led og kun et Chern-Simons-led i en klasse af teorier, har de begge de samme mangfoldigheder som deres modulimangfoldigheder af vortex-løsninger. Til sidst men ikke mindst, undersøger vi såkaldte fraktionale vortex-løsninger, som er understrukturer i energidensiteten i forskellige teorier, som er til stede enten på grund af deformationer eller singulariteter i målmangfoldigheden af solitonens effektive lav-energi-teori.

Preface

This Thesis is a final dissertation for my Ph.D. degree at the School of Graduate Studies Galileo Galilei (Scuola di Dottorato Galileo Galilei), University of Pisa with my academic supervisor Prof. Kenichi Konishi. The work presented here is based on the papers [1, 2, 3, 4, 5, 6, 7] and relevant publications related to the Thesis work are listed on a separate page before the Bibliography. Some of the mentioned papers are made in collaboration with my academic supervisor Prof. Kenichi Konishi and many collaborators: Dr. Roberto Auzzi, Dr. Stefano Bolognesi, Dr. Minoru Eto, Dr. Luca Ferretti, Dr. Toshiaki Fujimori, Dr. Takayuki Nagashima, Prof. Muneto Nitta, Dr. Keisuke Ohashi and Dr. Walter Vinci.

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Introductory tour

CHAPTER 0

Introduction and motivation

Solitons play an important role in a vast range of physics, from condensed matter physics, fluid dynamics through particle physics and cosmology to string theory. The most famous soliton, namely the vortex [11, 12] has been observed experimentally in type II superconductors in solid state physics, forming a so-called Abrikosov lattice.

In cosmology, strings, not of fundamental nature like the superstrings of string theory, but larger, very heavy strings – cosmic strings [13, 14, 15], are thought to be able to teach us about the physics of the early Universe, where large string webs explain the filaments and voids on the largest scales of the Universe, giving rise to density inhomogeneities. Unfortunately, a series of measurements (made by COBE and WMAP) in the cosmic microwave background (CMB) radiation exhibits so-called acoustic peaks, which rule out the cosmic strings as topological defects in the early Universe. These measurements however are well described within the so-called inflation models for cosmology.

In solid state physics, in particular in relation with the fractional quantum Hall effect, effective field theory descriptions of co-dimension 2, specifically including Chern-Simons interactions have proven especially important. This interaction changes the effective spin of a field depending on the coupling constant through the Aharonov-Bohm effect. An effective description of condensing electrons in a superconducting quantum Hall fluid can be described by a scalar field theory with some opportune Chern-Simons interaction.

One of the most important open problems in particle physics came along with the solution to the strong interactions, Quantum Chromodynamics (QCD), namely the asymptotic freedom yields good ultra-violet behavior, but on the contrary renders perturbation theory rather useless in the infra-red. However, this confinement of quarks is rather welcome as it explains why no free quarks have been observed. Nevertheless, in order to get back calculability, new methods are needed, that is, so-called non-perturbative methods are of necessity in this arena.

The most promising idea of a confinement mechanism is due to 't Hooft and Mandelstam which is a dual superconductor where the role of electric and magnetic charges are swapped. If then magnetic monopoles are condensed in the vacuum of QCD, the electric (color) sources are confined by flux tubes, i.e. vortices giving rise to linear Regge trajectories. These were in fact the reason for the earlier proposed dual models of strings describing the strong interacting sector (which however were discarded due to among other problems the tachyon in the spectrum). This idea however is not so easily realized and in fact it was first in supersymmetric ($\mathcal{N} = 2$) theories that this type of confinement mechanism was realized through the full solubility provided in Seiberg-Witten theory (however this is not QCD). One of the mayor breakthroughs in quantum field theory in the nineties, was indeed the seminal papers by Seiberg and Witten [16, 17] which in extended $\mathcal{N} = 2$ supersymmetric gauge theories found exact solutions of the low-energy effective theory. This was done exploiting the holomorphy of the superpotentials, non-renormalization theorems, the exact quantum corrections combined with the summation of the all-order contribution of instanton effects. This remarkable and enlightening result yielded not only a powerful method in field theory, the so-called Seiberg-Witten curve, but also demonstrated cleanly the ideas of electric-magnetic duality and confinement. This type of confinement found is however of an Abelian nature.

An interesting open problem in the context of confinement, where the vortex is a flux tube of color charges for condensed magnetic monopoles, is related to the quantization and duality due to Goddard, Nuyts, Olive and Weinberg (GNOW), who conjectured that the transformations of the monopoles are according to the dual of the unbroken gauge group. One of the remaining open problems in this relation are the difficulties that arise in quantizing the monopoles as there are non-normalizable zero-modes. Furthermore, there emerges a so-called topological obstruction in defining the unbroken gauge group for the monopoles. The non-Abelian vortex kicks in a good hope in this connection because of the calculability to a very high extent and therefore might shed light on the so far more obscure parts of this scenario, namely the monopoles. It is so far unknown which kind of monopoles would be relevant for a QCD-type confinement mechanism, that is, of Abelian or non-Abelian nature. All these open questions are further motives for studying these solitons of non-Abelian kind.

A final motivation for studying non-Abelian vortices is a vortex of a somewhat different kind, namely with non-Abelian first homotopy group. These are so-called quantized vortices [18, 19] which have been studied recently, exhibiting not only a non-trivial phase factor like normal vortices but combining it with a rotation of spin or orbital orientation. This type of vortices enjoys non-Abelian and non-commutative properties which are severely important, for instance, in collisions. Here so-called rung-vortices [20] can arise in many cases, which are deeply rooted in the non-Abelian nature. These systems with their vortices are especially interesting not only per se, but also due to the potential applications. Not only the theoretical part can be further studied but proposals for experiments are also possible here, for example in Bose-Einstein-Condensates (BECs), biaxial nematic liquid crystals and superconductors with high internal degrees of freedom.

With a list of motivations at hand, let us step back and describe what topological solitons are.

0.1 Topological solitons

A wave packet in some field that does not change its shape and intensity over time is called a solitary wave. If furthermore two solitary waves retain their shapes after a collision among them, they are denoted solitons. Sometimes both are called by the most popular name, solitons. Our interest lies in the class of solitons which have a topological origin. They will however not in general pass through each other with no change in velocities as they might slow down a bit and spit off some elementary quanta of light or other elementary particles.

Let us start with the simplest example of a static, topological soliton or topological defect, namely the domain wall. It belongs to a large class of solitons which are tightly associated with spontaneous symmetry breaking (SSB), which usually is generated by some scalar fields acquiring a vacuum expectation value (VEV). But it could also be due to the formation of some fermion condensate. The easiest example may be demonstrated in this real scalar field theory in $1+1$ dimensions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^2 - \xi\right)^2 \ , \quad \lambda, \xi > 0 \ , \tag{1}$$

which enjoys a $\mathbb{Z}_2$ symmetry $\phi \to -\phi$, which however is broken by the vacuum: $\mathbb{Z}_2 \to \mathbb{1}$. The scalar field can choose between two vacua, $\phi = -\sqrt{\xi}$ and $\phi = \sqrt{\xi}$, which both imply a vanishing energy density. Consider now a configuration that has the first vacuum $\phi = -\sqrt{\xi}$ at $x \to -\infty$ and the second vacuum $\phi = \sqrt{\xi}$ at $x \to \infty$. When ϕ has to traverse $\phi = 0$, it will create a non-zero energy density – a solitonic particle.

Most famous are the vortices which are supported by the first homotopy group of the vacuum manifold of a field theory: $\pi_1(\mathcal{M}) = \mathbb{Z}$. For a non-trivial second homotopy group $\pi_2(\mathcal{M}) = \mathbb{Z}$, monopoles and lumps can be supported while for the third one $\pi_3(\mathcal{M}) = \mathbb{Z}$ so-called Skyrmions (also denoted textures) have support. The first two have linear field theories as origin (though non-linear field equations), while the last has a non-linear one. In general, this type of defect is characterized by the l-th homotopy group $\pi_l(\mathcal{M})$ of the vacuum manifold $\mathcal{M}$. This gives us a soliton of co-dimension $l+1$ having a $d-l-2$ world-volume, where d is the number of space-time dimensions. Classifying instead in terms of co-dimensions, a (non-trivial) one dimensional field configuration interpolating two vacua of a field theory is a domain wall, while for two dimensions we are back to the vortex, for three dimensions it is a monopole and finally in four dimensions there is a possibility for an instanton.

An important question is about the stability of static solitons and in fact Derrick's no-go theorem [21] which is based on a scaling argument tells us that no finite energy field configuration in more than one spatial dimension, other than the vacuum, can have a stationary point, which means that no finite energy solitons in spatial dimensions $\hat{d} > 1$ are stable. Let us review this calculation briefly. Considering for simplicity a scalar field under rescaling $x \to \mu x, \phi(x) \to \phi(\mu x)$, then the derivative scales like $\partial_i \to \mu\partial_i$ while the integration measure picks up a Jacobian factor of $\mu^{-\hat{d}}$. Let us further assume that the field describes a non-trivial stationary solution. That is, it should be a critical configuration of the energy functional

$$E[\phi(x)] = \int_{\mathbb{R}^{\hat{d}}} d^{\hat{d}}x \ \left\{|\partial_i\phi(x)|^2 + V[\phi(x)]\right\} \ , \tag{2}$$

which means that

$$\left.\frac{dE[\phi(\mu x)]}{d\mu}\right|_{\mu=1} = 0 \ . \tag{3}$$

We then easily find that

$$E[\phi(\mu x)] = \int_{\mathbb{R}^{\hat{d}}} d^{\hat{d}}(\mu x) \ \left\{ \mu^{2-\hat{d}} |\partial_i \phi(\mu x)|^2 + \mu^{-\hat{d}} V[\phi(\mu x)] \right\} \ . \tag{4}$$

Combining the two Eqs. (3) and (4) leaves us with

$$(2 - \hat{d}) \int_{\mathbb{R}^{\hat{d}}} d^{\hat{d}}(\mu x) \ |\partial_i \phi(\mu x)|^2 = \hat{d} \int_{\mathbb{R}^{\hat{d}}} d^{\hat{d}}(\mu x) \ V[\phi(\mu x)] \ , \tag{5}$$

which gives us two possibilities. For $\hat{d} = 1$ we can have non-trivial stationary solutions, while for $\hat{d} = 2$ we can only have interesting solutions for $V = 0$ which are harmonic maps corresponding to σ model lumps. We will see that there are several ways to evade the obstacle, especially in the case of $\hat{d} = 2$ which will be the most studied case in this Thesis. Let us take one step further and consider the addition of a gauge field, we see that for making the covariant derivative scale covariantly $\mathcal{D}_\mu \to \mu \mathcal{D}_\mu$, the gauge field has to scale like $A_\mu \to \mu A_\mu$ and thus we have

$$E[\phi(\mu x)] = \int_{\mathbb{R}^{\hat{d}}} d^{\hat{d}}(\mu x) \ \left\{ \frac{1}{4g^2} \mu^{4-\hat{d}} F_{\mu\nu}^2(\mu x) + \mu^{2-\hat{d}} |\mathcal{D}_i \phi(\mu x)|^2 + \mu^{-\hat{d}} V[\phi(\mu x)] \right\} \ . \tag{6}$$

This shows clearly that for $\hat{d} = 2$, the gauge fields can balance the scaling due to the potential, thus giving rise to vortices. We will delve into those later. A final comment on the scaling arguments is the observation that in four spatial dimensions, the gauge field interactions do not scale

$$E[\phi(\mu x)] = \int_{\mathbb{R}^{\hat{d}}} d^{\hat{d}}(\mu x) \ \left\{ \frac{1}{4g^2} \mu^{4-\hat{d}} F_{\mu\nu}^2 \right\} \ , \tag{7}$$

and does indeed give rise to a soliton without any SSB – the instanton, which we will turn to next.

Indeed the most important soliton in high energy physics might be the instanton which was found by Belavin, Polyakov, Schwartz and Tyupkin (BPST) in Ref. [22] in pure $SU(2)$ Yang-Mills theory as a solution to the renowned self-dual equation. The Yang-Mills action can be written as

$$\begin{aligned} S &= -\frac{1}{2g^2} \mathrm{Tr} \int_{\mathbb{R}^4} F_{\mu\nu} F_{\mu\nu} \\ &= -\frac{1}{4g^2} \mathrm{Tr} \int_{\mathbb{R}^4} \left\{ (F_{\mu\nu} \mp {}^\star F_{\mu\nu})^2 \pm 2 F_{\mu\nu} {}^\star F_{\mu\nu} \right\} \ , \end{aligned} \tag{8}$$

which has a lower bound on the action given by $S = k(8\pi^2)/g^2$ where k is the instanton number. The bound is saturated when the self dual equation

$$F_{\mu\nu} = \pm {}^\star F_{\mu\nu} \ , \tag{9}$$

is satisfied. Instantons are somewhat different than the other topological defects and solitons that we have mentioned so far. There is no need for spontaneous symmetry breaking here.

So what is the topology argument? It is simply the boundary of space-time $\partial\mathbb{R}^4 \simeq S^3$ which is mapped to the gauge group $SU(2) \simeq S^3$. This is a well-known problem in topology, and the maps are characterized by $\pi_3(S^3) = \mathbb{Z} \ni k$. Hence the instant number is of a topological origin. An interesting comment in store is to note that the action is inversely proportional to the gauge coupling constant squared: $S \propto g^{-2}$, while the quantum amplitudes described by the partition function behave as $\mathcal{Z} \sim e^{-S}$. Hence the instantons become important at strong coupling. Instantons can be interpreted as particles in a $4+1$ dimensional space-time, where they are simply static soliton solutions to the self-dual equation (9). In $3+1$ dimensions, however, the formal instanton solution in Euclidean space-time, makes it not a particle in space, but an instantaneous event in space-time, hence the name instanton. This can seem like an odd and non-important technicality, but it turns out, as already mentioned that they dominate the path-integral, which makes ultimately sense by wick-rotating the physical action.

A remarkable solution to the self-dual equation was found by Atiyah, Drinfeld, Hitchin and Manin (ADHM) and commonly denoted the ADHM construction [23]. This solution provides a formal solution which is self-dual by construction and in the cases of $N \leq 3$ explicit solutions have been obtained [24, 25]. Finally, one of the most important implications of instantons probably is the resolution of the so-called $U(1)$ problem by 't Hooft [26], which via the axial current anomaly explains the excessively large mass of the η' particle, which otherwise is a would-be Goldstone boson.

Let us make a comment about supersymmetry. It turns out that the critical coupling of vortices, the boundary between type I and type II corresponds to solutions preserving some amount of supersymmetry and can be obtained by dimensional (Scherk-Schwarz) reduction from four-dimensional Euclidean instantons. Working with four dimensions, we have eight supercharges in $\mathcal{N}=2$ extended supersymmetric theories and the instantons preserve exactly half, hence the notion 1/2-BPS solitons. The same goes for monopoles, vortices and domain wall in these theories. Combining the solitons of different kinds, for instance vortices with domain walls yields 1/4 BPS composite objects [27, 28, 29, 30].

A final comment about instantons is their beautiful D-brane picture realization. In two words, D-branes are 1/2 BPS solitons in string theory on which fundamental strings (F-strings) can end on. It furthermore turns out that D-branes have gauge theories living on their world-volume (a $U(1)$ gauge theory to be precise) while a stack of N coincident D-branes have a $U(N)$ gauge theory on their world-volume. To this end, a D4-brane enjoys a $4+1$ dimensional gauge theory and we already mentioned that the instanton can be interpreted as a particle in exactly this "space-time". Here the particles are manifested in terms of D0-branes, which are point-like branes. It was discovered that the coupling of D0-branes to D4-branes provides exactly the moduli space of k instantons in the world-volume of the D4-branes, where k is simply the number of D0-branes [31, 32, 33] and exhibits $8k$ real parameters which are precisely the moduli of k instantons.

Let us close the book on instantons for now, leave the interested reader to the literature and move on to the most famous particle not (yet) discovered:[1] the magnetic monopole. The monopoles are in fact disallowed by Maxwell's equations which in turn is due to the Bianchi

[1] Let us define the Higgs to be discovered in the sense that the masses of the W-bosons have been measured, "only" the mass and constituent nature remain still a task for the Large Hadron Collider (LHC) to discover.

identity in classical electromagnetism. However, Dirac noticed that there is a striking duality between the electric equations of motion and the magnetic ones [34, 35]

$$\partial_\mu F^{\mu\nu} = -j^\mu \ , \quad \partial_\mu \tilde{F}^{\mu\nu} = -k^\mu \ , \quad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \ , \tag{10}$$

which are manifestly invariant under $(F, \tilde{F}) \to (\tilde{F}, -F)$ together with $(j, k) \to (k, -j)$. The monopoles in Abelian gauge theory named after Dirac are singular, however in non-Abelian gauge theories, non-singular monopoles were found by 't Hooft and Polyakov [36, 37, 38]. Especially, in an $SU(2)$ gauge theory with adjoint scalars fields, a symmetry breaking that gives rise to $\pi_2(SU(2)/U(1) \simeq S^2) = \mathbb{Z}$ does indeed support monopoles topologically. An important implication of the presence of just a single monopole is that suddenly all electric charges are quantized due to the relation

$$\frac{qQ}{4\pi} = n \in \mathbb{Z} \ , \tag{11}$$

where q is the electric charge while Q is the magnetic charge. The quantization gets more involved for higher-rank gauge groups and this is the already mentioned story of the GNOW duality.

An important development in the monopoles was due to Nahm [39], who constructed all the solutions to find the moduli space of monopoles. This solution bears the name the Nahm construction, which is somewhat similar to the ADHM construction for instantons. The moduli space of monopoles was then used by Manton [40], to describe adiabatic dynamics of multiple monopoles. For this reason it is denoted the moduli space approximation or the geodesic approximation, because the monopoles can simply be described by the low-energy effective theory on the moduli space of the monopoles. This technique has been adapted to many other soliton systems as well.

Another type of duality is also sometimes present in solitonic systems. Consider a generic theory with a SSB giving rise to a VEV $\langle\phi\rangle = \sqrt{\xi}$ with a potential coupling λ. Then the mass-squared of the scalar field is $\sim \lambda\sqrt{\xi}$, while the mass of the soliton is $\sqrt{\xi}$. Thus at weak coupling $\lambda \ll 1$, the soliton is much heavier than the elementary particle, while the situation is exactly opposite in the strong coupling regime. This is a hint of some of the importance of solitons in gauge theories. A prime example was found between the sine-Gordon model and the massive Thirring model in $1 + 1$ dimensions by Coleman [41].

Finally, an electromagnetic duality has been conjectured in gauge field theories by Montonen and Olive [42]. On one side of the duality there is a perturbative spectrum of elementary gauge particles of an unbroken gauge group, while on the other side of the duality there are dual gauge fields where monopoles are in the perturbative spectrum. Not only do the Noether currents get interchanged with the topological currents but the coupling constant gets interchanged with its inverse. In $\mathcal{N} = 4$ super Yang-Mills theory this duality has been checked to some extent in Refs. [43, 44, 45, 46], for instance by reducing S-duality to T-duality. In string theory [47] the picture of the electromagnetic duality is somewhat more intuitive, where the duality generically is denoted S-duality which acts on the complex coupling constant $\tau = a + ig_s^{-1}$, where a is the VEV of a massless Ramond-Ramond scalar and g_s is the string coupling constant. In this context, the $\mathcal{N} = 4$ super Yang-Mills theory lives on the world-volume of N D3-branes on which both D1-branes (D-strings) and F-strings can end. Here the F-strings represent electrically charged gauge particles while D-strings are monopoles. These are interchanged under S-duality while the coupling is inverted. It is intriguing (a dream of some theoretical physicists) that in some models, weakly and strong coupled theories are mutually related by a duality.

Now we will take a breath from all the strings of fundamental kind and step back to what will be our main concern in this Thesis, namely magnetic flux tubes – vortices.

0.2 Vortices

Considering now two spatial dimensions, there are at least three ways around the at first disappointing no-go theorem by Derrick: stabilizing the configuration with some flux; adding angular momentum (which leads to so-called Q-lumps); adding higher derivative terms (which leads to so-called baby-Skyrmions). We stick to the first resolution to the problem and add a flux, i.e. we shall consider a gauge theory. Taking a local $U(1)$ gauge theory and breaking completely the gauge symmetry in the vacuum: $U(1) \to \mathbb{1}$, we have the topological support

$$\pi_1\left(U(1)\right) \simeq \pi_1\left(S^1\right) = \mathbb{Z} \ni k \ , \tag{12}$$

which is the basics of the Abrikosov-Nielsen-Olesen (ANO) vortex. To understand this support, it is useful to think about what happens pictorially. The soliton solution is time independent, and we can think of just an xy-plane with a scalar field configuration. The potential is broken from being in a symmetric phase to enter an asymmetric phase. This is done with the Higgs mechanism or in supersymmetric theories we say adding an Fayet-Iliopoulos parameter $\xi > 0$, which puts the theory *on the Higgs branch*. To consider a simple example we can think about

$$V = \frac{\lambda}{2}\left(|\phi|^2 - \xi\right)^2 \ , \tag{13}$$

which is a $\lambda|\phi|^4$ potential possessing spontaneous symmetry breaking. The soliton support comes from a non-trivial mapping that maps the spatial infinity of the field configuration onto the vacuum manifold, see Fig. 1. Clearly the field cannot change its magnitude at infinity and in the same time belong to a class of *finite energy configurations*. The phase however can turn $k \in \mathbb{Z}$ times. This wrapping on a circle called vorticity, cannot be unwrapped with a finite energy and the soliton is thus (classically) stable.

Extending this type of vortex configuration, we will start by considering a larger symmetry group, say $SU(N)$ but still only gauge a $U(1)$ subgroup (we say we add flavors to the theory). This type of model is termed semi-local, because only some of the symmetry is gauged. The symmetry breaking this time however is not supported by a non-trivial first homotopy group as

$$\pi_1\left(\frac{SU(N)}{U(1)\times SU(N-1)}\right) = \mathbb{1} \ , \tag{14}$$

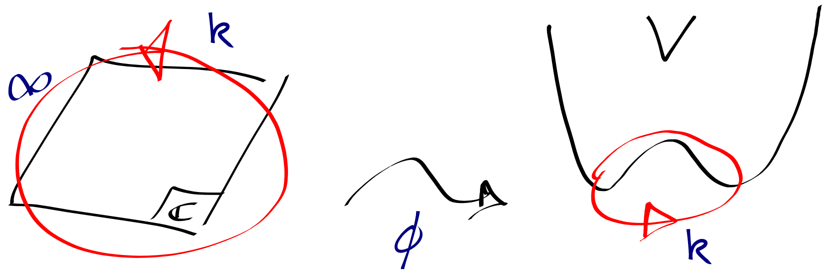


Figure 1: Sketch of a mapping from spatial infinity of a scalar field configuration onto the vacuum manifold. This is characterized by the first homotopy group π_1.

however, the local part is still the same, hence some stability is expected, which turns out to hold; the result of Ref. [48] shows that for type I superconductivity ($\beta < 1$) the embedding of a local (ANO) vortex is indeed stable. Considering the second homotopy group

$$\pi_2\left(\frac{SU(N)}{U(1)\times SU(N-1)}\right) = \mathbb{Z} \ , \tag{15}$$

thus provides topological stability, however this second homotopy group gives support to so-called lump solutions. Lumps can be thought of as vortices at very strong gauge coupling or very low energy. For these semi-local vortices it works quite well, however, for local (ANO) vortices, the size of the vortex goes as the inverse gauge coupling constant and leaves behind only a delta-spike. In the associated non-linear σ model (NLσM), these are denoted small-lump singularities (where the name is somewhat inspired from small instanton singularities).

In the gauge theories under consideration, a potential is necessary to break the symmetry spontaneously and the gauge fields are needed to stabilize the interpolating solutions. There will be a particular coupling of the potential where the equations of motion will decrease in order from second to first order differential equations. This turns out to be due to an enhanced symmetry, namely supersymmetry, which gets restored at this point. Rewriting the energy functional in terms of a sum of squares with just a surface term and a term being the magnetic flux is called the Bogomol'nyi completion (or sometimes Bogomol'nyi trick) and the first order equations are called BPS-equations (due to Bogomol'nyi-Prasad-Sommerfield) for when obeyed, the system saturates the Bogomol'nyi bound. For the ANO vortex the coupling of the quartic potential in the Abelian-Higgs model has three types of vortices: for $\beta > 1$ ($\beta < 1$) it yields type II (I) vortices, while for $\beta = 1$ it has BPS vortices. A more physical explanation for this critically coupled theory is that the force due to the scalar field exactly cancels the force due to the magnetic field. Thus the net force between critically coupled (BPS) vortices vanishes. This kind of cancellation often takes place in supersymmetric field theories.

Along the lines of the extension to the semi-local vortex we could consider gauging the full $SU(N)$ symmetry. This does not support the vortex of the same kind, instead

$$\pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N \ , \tag{16}$$

which is denoted a $\mathbb{Z}_N$-string. The trick to obtain the, what has been shown to be, the interesting type of non-Abelian vortices, is to include an overall $U(1)$ factor, for example considering a $U(N)$ gauge theory with the symmetry breaking pattern $U(N) \to \mathbb{1}$:

$$\pi_1(U(N)) = \mathbb{Z} \,. \tag{17}$$

In analogy with the Abelian ANO vortices, the system can be BPS saturated for a particular coupling of the potential which is a point of special interest. Namely a moduli space of solutions – all with the same tension – is present in the theory and for a single non-Abelian vortex in this particular $U(N)$ model (which enjoys $\mathcal{N} = 2$ supersymmetry) turns out to have a moduli space being $\mathbb{C}P^{N-1} \simeq SU(N)/[SU(N-1) \times U(1)]$.

Until now we had four dimensional spacetime in mind, however if we relax this constraint and step down to three dimensions, two immediate things happen: the renormalizable potential can now be of sixth order and the gauge fields can be controlled by a Chern-Simons interaction (which is possible only in odd dimensions). The Chern-Simons term is of topological origin, viz. there is no contraction with the metric. The vortex constructed in a theory with a Chern-Simons kinetic term is a “dyonic” configuration having electric charge attached to its magnetic flux and moreover can alter its spin and angular momentum by changing the Chern-Simons coupling constant. There are now three possibilities, having a Maxwell kinetic term, a Chern-Simons term or both. Having both the terms gives rise to a complex system which can interpolate between Chern-Simons vortices and ANO vortices. We will come back to the Chern-Simons theories in the course of the next Chapter, where we will construct the mentioned kinds of vortex in much more detail.

0.3 Kähler and hyper-Kähler quotients

An important technology that we will use to construct approximate solutions for semi-local vortices (viz. lumps) is the Kähler quotient construction. This fact is intimately related to that strong gauge coupling or long distance correspond to limits which are well-described by a low-energy effective theory where the vector multiplet, the gauge fields, have been integrated out. In the following we will only consider supersymmetric gauge theories. To be precise, the Higgs branch of $\mathcal{N} = 2$ supersymmetric QCD is hyper-Kähler and the low-energy effective theory on this Higgs branch is described by an $\mathcal{N} = 2$ non-linear σ model (NLσM) on the hyper-Kähler manifold [17, 49, 50].

The target space of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric NLσMs, with four and eight supercharges, was in fact studied much earlier and it was shown to be Kähler [51] and hyper-Kähler [52], respectively. The notion of the hyper-Kähler quotient was first found in physics in the Refs. [53, 54, 55, 56], but was later formulated mathematically by Hitchin, Karlhede, Linström and Rǒcek [57] (we recommend Ref. [58] as a review for physicists). It was found that a $U(1)$ hyper-Kähler quotient [53, 54, 55] recovers the Calabi metric [59] on the cotangent bundle over the projective space, $T^\star\mathbb{C}P^{N-1}$, while its $U(N)$ generalization leads to the cotangent bundle over the complex Grassmann manifold, $T^\star G_{N,N}$ [56]. The hyper-Kähler manifolds also appear in the moduli spaces of Bogomol’nyi-Prasad-Sommerfield (BPS) solitons such as Yang-Mills instantons [23, 24, 60, 61] and BPS monopoles [62, 63]. The hyper-Kähler quotient offers a powerful tool to construct these hyper-Kähler manifolds: instanton moduli spaces [23] and monopole moduli spaces [64]. Gravitational instantons

[65, 66], Yang-Mills instantons on gravitational instantons [67, 68] and toric hyper-Kähler manifolds [69] are all constructed using the hyper-Kähler quotient.

Let us step back to our gauge theories. In the cases of an $SU(N)$ or a $U(N)$ gauge theory with hypermultiplets charged commonly under $U(1)$, the metrics on the Higgs branch and their Kähler potentials are known explicitly and are given by the Lindström-Roček metric [56]. Another example is the $U(1) \times U(1)$ gauge theory with three hypermultiplets of certain charges which gives the space: $T^\star F_n$ with F_n being the Hirzebruch surface [70]. The Higgs branches of quiver gauge theories are known to be gravitational instantons and Yang-Mills instantons on gravitational instantons [65, 67, 68].

A lump solution was first found in the $O(3)$ σ model, or the $\mathbb{C}P^1$ model by Polyakov and Belavin [71]. It was then later generalized to the $\mathbb{C}P^n$ model [72, 73, 74], the Grassmann model [75, 76, 77], and other Kähler coset spaces [78, 79]. Lumps are topological solitons associated with $\pi_2(\mathcal{M})$ with $\mathcal{M}$ being the target Kähler manifold. Their energy saturates the BPS bound of the topological charge written as the Kähler form of $\mathcal{M}$ pulled-back to the two-dimensional space.[2] The lump solutions preserve half of supersymmetry, when embedded into supersymmetric theories. The dynamics of lumps was studied [82, 83] by the already mentioned moduli space (geodesic) approximation. Lumps are related to vortices in gauge theories as follows. $U(1)$ gauge theories coupled to several Higgs fields often admit semi-local vortex-strings [84, 85]. In the strong gauge coupling limit, gauge theories reduce to NLσMs whose target space is the moduli space of vacua in the gauge theories, and in this limit, semi-local strings reduce to lump-strings. For instance, a $U(1)$ gauge theory coupled to two charged Higgs fields reduces to the $\mathbb{C}P^1$ model, while the semi-local vortex-strings in Refs. [84, 85] reduce to the $\mathbb{C}P^1$ lumps [86, 48, 87]. In the gauge theories at finite coupling, the large distance behavior of semi-local strings is well approximated by lump solutions. The sizes or widths of semi-local strings are moduli of the solution in the BPS limit, and accordingly, the lumps also possess size moduli. When the size modulus of a semi-local string vanishes, the solution reduces to the Abrikosov-Nielsen-Olesen (ANO) vortex [11, 12] which is called a local vortex. This limit corresponds to a singular configuration in the NLσM, which is called the small lump singularity. Lumps and semi-local strings are also candidates of cosmic strings, see e.g. Refs. [88, 89, 90, 91, 92, 93], and appear also in recent studies of D-brane inflation etc. [94, 95, 96].

0.4 A brief overview of the related literature

The non-Abelian vortex, which here means the vortex solution which exhibits non-Abelian color-flux and orientational modes, was discovered by the two groups, MIT [9] and Pisa [97], independently. The solutions were made in unitary groups $G = U(N)$, where the crucial difference, with respect to the previously found $\mathbb{Z}_N$ strings, lies simply in the overall $U(1)$ factor giving rise to topological stability of the type $\pi_1(G) = \mathbb{Z}$. In the seminal paper [9], a brane construction was given and the moduli space was found by use of the Kähler quotient construction in the case of a single vortex or k separated vortices and furthermore,

[2] In the case of hyper-Kähler manifolds there exists a triplet of complex structures and Kähler forms. Accordingly, it has recently been found that there exists a BPS bound written by the sum of three different Kähler forms to three different planes in the three dimensional space [80, 81, 30].

the number of bosonic zero-modes were counted by means of index theorem methods. In the mean time the Tokyo group has been studying domain walls and junctions thereof in $\mathcal{N} = 1, 2$ supersymmetric theories and shortly after the discovery of the non-Abelian vortex, they found a systematic formalism to uncover the full moduli space of domain walls [98, 29, 99] which soon after was adapted to the non-Abelian vortices [100] and thus also in that case, they could systematically find all the moduli of the system. Around the same time non-Abelian vortices were also studied in six-dimensional space-time [101]. Also instantons were studied in the Higgs phase [102].

The problems of non-Abelian monopoles and the non-Abelian generalization of a dual superconductor describing a confinement mechanism were studied in Refs. [103, 104, 105].

A duality between two and four dimensions was found through the non-Abelian vortices [106], which explains the matching of the BPS-spectra found much earlier by Nick Dorey in these two different theories [107]. The two-dimensional theory on the vortex world-sheet, which in the case of the $U(2)$ group is a $\mathbb{C}P^1$ σ model, exhibits also kinks, which turn out to be Abelian monopoles in the four-dimensional theory [8]. Later it was found in the mass deformed $\mathcal{N} = 2$ theory, that a supersymmetry emergence takes place [108], giving rise to exactly this kind of kink-monopole on the world-sheet of the vortex, having two supersymmetries whereas in the bulk theory the monopole does not have any central charge at all.

The moduli space of two coincident non-Abelian $U(2)$ vortices, which is important also in the reconnection of cosmic strings, was studied in Ref. [109] by string-theory techniques and in Refs. [110, 111] by field theory techniques which differ by a discrete quotient giving rise to a conifold singularity in the moduli space. A demonstration that reconnection of non-Abelian strings of local or semi-local type is indeed universal was made in Ref. [112].

Manifestly supersymmetric effective Lagrangians were studied with the field of the moduli matrix formalism directly present in the Lagrangian density [113] and a duality between non-Abelian vortices and domain walls was found in Ref. [114].

In models with unitary gauge groups, adding additional flavors of squark matter to the theory, that is, considering the number of flavors greater than the number of colors, gives rise to so-called semi-local zero-modes which by their nature are non-renormalizable. This type of non-Abelian vortices was studied in Refs. [115, 116].

A superconformal non-Abelian vortex string was also studied in Ref. [117] where it is shown that the theory flows into an infra-red superconformal fixed point at low energies.

In connection with studying the dual models of confinement, the choice of different gauge groups is interesting and could give important hints about this fascinating topic. This gave rise to line of research investigating non-Abelian vortices with especially $SO(N)$ gauge groups [118], generic gauge groups [6] and in very much detail $SO(N), USp(2M)$ [3]. In this connection, the Kähler quotient and hyper-Kähler quotient for studying the gauge theories as well as the corresponding lumps as effective low-energy approximation to the vortices was studied in detail in Ref. [5]. This also gave rise to questions about the stability of semi-local non-Abelian vortices [119], as well as the impact of non-BPS corrections on the BPS vortex systems and interactions [120, 121].

A calculation of the partition function of statistical mechanics for non-Abelian BPS vortices on a torus was studied in Ref. [122]. The coupling of the non-Abelian strings with gravity was also studied in Ref. [123].

The next big program was taking the non-Abelian vortices to $\mathcal{N}=1$ supersymmetric theories. In $\mathcal{N}=2$ theories, the world-sheet of the vortex exhibits $\mathcal{N}=(2,2)$ supersymmetry together with a lot of other properties like the mentioned matching of the BPS-spectra of the two theories. This turns out also to be the case for the vortex string in $\mathcal{N}=1$ theories which are obtained by adding a superpotential with masses for the adjoint field, giving rise to the so-called heterotic strings, which were shown in Ref. [124] to have $\mathcal{N}=(0,2)$ supersymmetry. Soon after, several papers discussed the quantum properties, the relation between the bulk and world-sheet theory for various values of the deformation parameters as well as a large-N solution [125, 126, 127, 128].

Another direction is the non-Abelian vortex in the Higgs vacuum of $\mathcal{N}=1^\star$ theory with the unitary gauge group $SU(N)$, which is a mass deformation of the conformal $\mathcal{N}=4$ theory. It was studied for two colors in Ref. [129] and for larger N in Ref. [130] both in the weakly coupled field theory and in the IIB string dual (i.e. in the Polchinski-Strassler background [131]). The quantum phases of this theory has also been studied recently in Ref. [132].

The questions about dynamical Abelianization were addressed in Ref. [133].

Another formalism for describing the full moduli space of non-Abelian vortices on Riemann surfaces was given in Ref. [134], where the matrix describing the k vortices is factorized.

For the reader which is not familiar with this literature, there are several comprehensive reviews on the topic of non-Abelian vortices [135, 136, 137, 138].

Let us conclude this Section with mentioning the literature concerned with non-Abelian Chern-Simons vortices. The first studies of non-Abelian Chern-Simons vortices are made with a simple group, viz. $SU(2)$ and $SU(N)$ with fields in the adjoint representation [139, 140, 141] and later numerical solutions have been found [142]. In Refs. [143, 144] the non-Abelian Chern-Simons vortices have been studied with a $U(N)$ gauge group allowing for orientational modes to be present and they identified the moduli space of a single vortex solution. Furthermore, Refs. [145, 146, 147] have considered combining the Yang-Mills and the non-Abelian Chern-Simons terms for $U(N)$ gauge groups. In Ref. [145] the dynamics of the vortices has been studied and in Ref. [146] in addition to the topological charge, conserved Noether charges associated with a $U(1)^{N-1}$ flavor symmetry of the theory due to inclusion of a mass term for the squarks, were considered. In Ref. [147] numerical solutions have been provided.

Many related topics can be found in the excellent reviews [148, 149].

0.5 How to read this Thesis?

The organization of this Thesis is as follows. The next Chapter will present all the tools and concepts needed, developed and used in the rest of the Thesis, while the results of more detailed nature are presented separately in subsequent Chapters. This Chapter includes the results of the paper [6]. In Chap. 2, we will present the type III vortex solution in the Abelian non-BPS Chern-Simons-Higgs model. In Chap. 3, a phase transition between Chern-Simons type and ANO vortices in the limit of large magnetic flux is discussed. These two Chapters are based on the paper [7]. In Part III, we will first present a result on the quantization of the non-Abelian flux of the vortices à la GNOW, in Chap. 4. Next, results on the moduli space transition functions between various patches etc., of the local and semi-local $SO(N)$ and $USp(2M)$ vortices are presented in Chaps. 5, 6. In Chap. 7, we will present an index theorem providing the number of moduli in a generic non-Abelian vortex configuration of the type $U(1) \times G'$, with G' being a simple group. The these four Chapters are based on the big paper [3]. In Chap. 8, we will present the Kähler and hyper-Kähler quotient construction in $SO(2M)$ and $USp(2M)$ theories and their corresponding lump solutions in Chap. 9. These two Chapters are based on the paper [5]. In the paper [5], lumps of fractional nature were discovered and this topic is discussed in more detail in Chap. 10, which is based on the paper [2]. In Chap. 11, results on the non-Abelian Chern-Simons vortices with generic gauge groups are presented. This Chapter is based on the paper [1]. Finally, in Chap. 12, future directions are discussed.

CHAPTER 1

Setup and basics

In this Chapter we will go through all the key subjects of this Thesis, starting with the introduction of $\mathcal{N} = 2$ supersymmetry and its breaking to $\mathcal{N} = 1$ to arrive at the main model we will study (in Sec. 1.2), where we will explain the main ingredients in the construction of non-Abelian vortices. In relation with the strong coupling limit, we will introduce the machinery of Kähler quotients to obtain low-energy effective theories of corresponding supersymmetric gauge theories. Then we will reduce the theory to an Abelian theory and discuss non-BPS potentials and integrability. Finally, we will reduce the number of dimensions to $2 + 1$ and discuss Chern-Simons terms and their consequences. This Chapter will give a brief account for all the topics discussed in this Thesis, while all the results will be presented in the subsequent parts.

1.1 Supersymmetry

Let us start by introducing the main model that we will work with in different forms through out this Thesis. Let us consider an $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theory (i.e. having eight supercharges) consisting of an $\mathcal{N} = 2$ chiral multiplet in the adjoint representation of the gauge group G along with $N_{\rm F}$ hypermultiplets transforming under the fundamental representation of the gauge group. Let us build up the theory step-by-step. A manifestly invariant $\mathcal{N} = 2$ supersymmetric theory can be written in terms of a holomorphic prepotential. In our simple case of Yang-Mills we have the classical prepotential on the following form

$$\mathcal{F}(\Psi) = \frac{\tau}{2}\Psi^2 \,, \quad \tau = \frac{\theta_{\rm vac}}{2\pi} + \frac{4\pi i}{g^2} \,, \tag{1.1}$$

with $\theta_{\rm vac}$ being the vacuum angle and g is the coupling constant of super-Yang-Mills while the $\mathcal{N} = 2$ chiral superfield Ψ is constructed of $\mathcal{N} = 1$ superfields as follows[1]

$$\Psi = \Phi(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y}, \theta) + \tilde{\theta}^2 G(\tilde{y}, \theta) \,, \tag{1.2}$$

[1]We will use the standard notation of Wess and Bagger [150] when using superspace, however with the gauge coupling rescaled in front of the gauge kinetic term.

where $\tilde{\theta}$ are $\mathcal{N} = 2$ superspace (Grassmann) variables, $\tilde{y}^\mu = y^\mu + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}}$ is the $\mathcal{N} = 2$ variables while $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ are the $\mathcal{N} = 1$ combined variables and the $\mathcal{N} = 1$ superfields are

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \ , \tag{1.3}$$

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(y) + i\theta^2\bar{\theta}\bar{\lambda}(\bar{y}) - i\bar{\theta}^2\theta\lambda(y) + \frac{1}{2}\theta^2\bar{\theta}^2 D(y) \ , \tag{1.4}$$

$$W_\alpha = \frac{1}{8}\bar{\mathcal{D}}^2\left(e^{2V}\mathcal{D}_\alpha e^{-2V}\right) \ , \tag{1.5}$$

where Φ is an $\mathcal{N} = 1$ chiral superfield, V is the vector superfield in Wess-Zumino gauge, W_α is the super field strength and θ is the normal $\mathcal{N} = 1$ superspace (Grassmann) variable. The function G in Eq. (1.2) is defined as follows

$$G(\tilde{y},\theta) = \int d^2\bar{\theta}\ \Phi^\dagger\big(\tilde{y} - i\theta\sigma\bar{\theta},\theta,\bar{\theta}\big)\ e^{-2V\left(\tilde{y}-i\theta\sigma\bar{\theta},\theta,\bar{\theta}\right)} \ . \tag{1.6}$$

The Lagrangian density can now neatly be expressed as

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Tr}\ \Im\int d^2\theta d^2\tilde{\theta}\ \mathcal{F} \ , \tag{1.7}$$

which when integrated over $\tilde{\theta}$ yields the gauge part of the $\mathcal{N} = 2$ theory

$$\mathcal{L}_{\mathrm{SYM}} = \mathrm{Tr}\ \Im\left[\frac{\tau}{4\pi}\left(\frac{1}{2}\int d^2\theta\ W^\alpha W_\alpha + \int d^4\theta\ \Phi^\dagger e^{-2V}\Phi\right)\right] \ . \tag{1.8}$$

Now we have just a pure supersymmetric gauge theory, and we see that it is simply the correct combination of an $\mathcal{N} = 1$ gauge multiplet together with an $\mathcal{N} = 1$ chiral multiplet, however we need still to add matter multiplets. These are $\mathcal{N} = 2$ hypermultiplets and we will add N_{F} of them with the following superpotential dictated by $\mathcal{N} = 2$ supersymmetry

$$\mathcal{W} = \sqrt{2}\ \mathrm{Tr}\ \tilde{Q}\Phi Q + \sum_{i=1}^{N_{\mathrm{F}}} m_i\tilde{Q}_i Q^i \ , \tag{1.9}$$

while their kinetic terms are given by the following Kähler potential

$$K = \mathrm{Tr}\left[Q^\dagger e^{-2V}Q + \tilde{Q}e^{2V}\tilde{Q}^\dagger\right] \ . \tag{1.10}$$

The matter superfields Q are all in the fundamental representation while $\tilde{Q}$ are in the anti-fundamental representation. The superfields Φ and V transform under the adjoint representation of the gauge group. The Lagrangian density for the hypermultiplets is

$$\mathcal{L}_{\mathrm{hyper}} = \int d^4\theta\ K + \int d^2\theta\ \mathcal{W} + \int d^2\bar{\theta}\ \bar{\mathcal{W}} \ . \tag{1.11}$$

Finally, we can collect the pieces and write the Lagrangian for the theory

$$\mathcal{L}_{\mathcal{N}=2} = \mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mathrm{hyper}} \ . \tag{1.12}$$

We still have the possibility to add a Fayet-Iliopoulos (FI) term to the theory *without* breaking $\mathcal{N} = 2$ supersymmetry. Introducing a triplet of $SU(2)_R$ R-symmetry which is present (and manifest in the $\mathcal{N} = 2$ superspace notation)

$$\frac{1}{\sqrt{2}}\left(F_1 + iF_2\right) \equiv F\ , \quad F_3 \equiv D\ , \tag{1.13}$$

we can add the following term to the Lagrangian density

$$-\sum_{i=1}^{3} \xi_i F_i\ , \tag{1.14}$$

which clarifies the equivalence of introducing the FI parameter with a D-term or with an F-term [151, 152, 153].

1.1.1 Breaking supersymmetry and generating an FI parameter

After having built up this neat non-Abelian $\mathcal{N} = 2$ gauge theory, let us break it down, at least to $\mathcal{N} = 1$ supersymmetry. We can do it by adding a deformation of the form of an $\mathcal{N} = 2$ non-invariant superpotential

$$\mathcal{W}_{\mathcal{N}=1} = \mu \ \mathrm{Tr}\ \Phi^2\ . \tag{1.15}$$

Truncating the chiral superfield to linear order still renders the theory $\mathcal{N} = 2$ invariant, however keeping all the terms generally breaks $\mathcal{N} = 2$. In the literature this is denoted a softly broken $\mathcal{N} = 2$ (down to $\mathcal{N} = 1$) model. If we keep the linear order in the superpotential, the term will act as an FI F-term parameter.

In general we will think of a symmetry breaking of the form

$$H \quad \xrightarrow{m} \quad G \quad \xrightarrow{\mu m} \quad \mathbb{1}\ . \tag{1.16}$$

There are a number of different non-Abelian vacua which are termed r-vacua [154, 155, 156], where r denotes the rank plus one of the unbroken gauge symmetry. Considering for instance $SU(N+1)$, we can choose the $r = N$ vacuum by the following VEV of the adjoint scalar

$$\langle\phi\rangle = \mathrm{diag}\left(m\,\mathbf{1}_N, -Nm\right)\ . \tag{1.17}$$

When $\mu \neq 0$ this will induce the above mentioned FI F-term parameter $\propto \mu m$ with the gauge symmetry breaking in this case

$$SU(N+1) \quad \xrightarrow{m} \quad \frac{U(1)\times SU(N)}{\mathbb{Z}_N} \quad \xrightarrow{\mu m} \quad \mathbb{1}\ . \tag{1.18}$$

This type of system is in particular interesting in the context of non-Abelian monopoles which are attached to the non-Abelian vortex [103, 97]. In that case the first symmetry breaking gives rise to the non-Abelian monopoles and possesses topologically supported stability via the second homotopy group

$$\pi_2\left(H/G\right) \neq \mathbb{1} \ , \tag{1.19}$$

while the vortices arise from the second breaking and are supported topologically by the first homotopy group

$$\pi_1\left(G\right) \neq \mathbb{1} \ . \tag{1.20}$$

We have demonstrated this way of naturally generating our model from a simple gauge group with a well-behaved theory at high energies, which at low energies has the $U(1)$ factor of crucial importance for constructing the vortices and it naturally has a symmetry breaking potential. Now that we know that our model can naturally be embedded at higher energies, we can forget about the high-energy theory with the associated monopoles, and work directly with the low-energy theory with a D-term FI parameter (and not the F-term type parameter like in the above described scenario).

1.1.2 Component fields

Now we are in shape to write down the Lagrangian in terms of bosonic component fields, which will be the basis of our further analyses in this Thesis. As mentioned in the last Section, we need to consider the complete breaking of the gauge symmetry to construct the vortices. Thus we write the theory directly with gauge group G. The Lagrangian density reads

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\bigg[&- \frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2g^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \frac{2}{e^2} \partial_\mu \phi \left(\partial^\mu \phi\right)^\dagger + \frac{2}{g^2} \mathcal{D}_\mu \hat{\phi} \big(\partial^\mu \hat{\phi}\big)^\dagger \\ &+ \mathcal{D}_\mu q \left(\mathcal{D}^\mu q\right)^\dagger + \left(\mathcal{D}_\mu \tilde{q}\right)^\dagger \mathcal{D}^\mu \tilde{q} \bigg] - V_D - V_F \ ,\end{aligned} \tag{1.21}$$

where we have explicitly differentiated between the Abelian gauge symmetry which now has the self coupling e while the non-Abelian gauge coupling is denoted by g. The generators of the gauge group $G/U(1)$ are all normalized as

$$\mathrm{Tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab} \ , \tag{1.22}$$

where the indices a, b are color indices and run from $a = 1, \ldots, \dim(G) - 1$. Also the Abelian generator has the same normalization

$$t^0 = \frac{\mathbf{1}_N}{\sqrt{2N}} \ . \tag{1.23}$$

The field strength of Abelian and non-Abelian kind are defined as follows

$$F_{\mu\nu} \equiv F^0_{\mu\nu} t^0 \ , \quad F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu \ , \tag{1.24}$$

$$\hat{F}_{\mu\nu} \equiv F^a_{\mu\nu} t^a = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right] \ , \quad A_\mu \equiv A^a_\mu t^a \ , \tag{1.25}$$

whereas the adjoint scalar fields are defined as follows

$$\phi \equiv \phi^0 t^0 \ , \quad \hat{\phi} \equiv \phi^a t^a \ . \tag{1.26}$$

Finally, we have the fundamental squark fields q and their anti-fundamental counter parts $\tilde{q}$ which are written in matrix notation as ${q_a}^f$ with a being the color index while f is the flavor index. The anti-fundamentals on the other hand are defined as ${\tilde{q}_f}^{\ a}$. The flavor indices run over $N_{\rm F}$ multiplets and are conveniently summed in the matrix product.

The potential receives contributions from the D-terms

$$V_D = \frac{g^2}{2}\mathrm{Tr}\left(\frac{i}{g^2}f^{abc}(\phi^\dagger)^b\phi^c + qq^\dagger t^a - \tilde{q}^\dagger\tilde{q}t^a\right)^2 + \frac{e^2}{2}\mathrm{Tr}\left(qq^\dagger t^0 - \tilde{q}^\dagger\tilde{q}t^0\right)^2 \ , \tag{1.27}$$

while the F-terms give rise to

$$\begin{aligned} V_F = {} & 2g^2\mathrm{Tr}\left|q\tilde{q}t^a\right|^2 + 2e^2\mathrm{Tr}\left|q\tilde{q}t^0 - \xi\right|^2 \\ & + 2\sum_{f=1}^{N_{\rm F}}\left|\left(\phi + \hat{\phi} + \frac{1}{\sqrt{2}}m_f\right)q^f\right|^2 + 2\sum_{f=1}^{N_{\rm F}}\left|\left(\phi + \hat{\phi} + \frac{1}{\sqrt{2}}m_f\right)(\tilde{q}^\dagger)^f\right|^2 \ , \end{aligned} \tag{1.28}$$

where the FI (F-term) parameter is given by $\xi = \mu m/\sqrt{2}$.

At the classical level there is a unique vacuum in the case of $G = U(N)$ theories, however it is not quite so in general, for instance for $G = U(1) \times SO(N)$ or $G = U(1) \times USp(2M)$ theories. There is however a unique characteristic about the vacuum of the $G = U(N)$ theories, which is an unbroken global symmetry, which is a combination of the global part of the color transformations and the flavor symmetry. That is, in the vacuum

$$\langle q \rangle = \langle \tilde{q}^\dagger \rangle = \sqrt{\frac{\xi}{N}}\mathbf{1}_N \ , \quad \langle \phi + \hat{\phi} \rangle = 0 \ , \tag{1.29}$$

there exists the following global color-flavor symmetry

$$\left\{q, \tilde{q}^\dagger\right\} \to U_c\left\{q, \tilde{q}^\dagger\right\}U_f^\dagger \ , \quad \left\{\phi, \hat{\phi}, F_{\mu\nu}, \hat{F}_{\mu\nu}\right\} \to U_c\left\{\phi, \hat{\phi}, F_{\mu\nu}, \hat{F}_{\mu\nu}\right\}U_c^\dagger \ . \tag{1.30}$$

All these fields are not really needed to construct the vortex solutions which will be our main interest in this Thesis, so we will simplify the theory a bit by setting $\tilde{q} = q^\dagger$ and $\phi = \hat{\phi} = 0$. This leads us to the main model that we will study. In the next Section we will construct solutions explicitly and completely general, leaving the possibility for a generic gauge group of the kind

$$G = U(1) \times G' \ , \tag{1.31}$$

with G' being a simple (non-Abelian) gauge group.

1.2 Vortex construction and moduli matrix formalism

We focus our attention on the classical Lie groups $G' = SU(N), SO(N)$ and $USp(2M)$, leaving the exceptional groups to a short discussion in Sec. 1.2.2. For $G' = SO(N)$, $USp(2M)$ their group elements are embedded into $SU(N)$ ($N = 2M$ for USp) by constraints of the form, $U^{\mathrm{T}} J U = J$, where J is the rank-2 invariant tensor

$$J = \begin{pmatrix} \mathbf{0}_M & \mathbf{1}_M \\ \epsilon \mathbf{1}_M & \mathbf{0}_M \end{pmatrix} , \quad J_{SO(2M+1)} = \begin{pmatrix} \mathbf{0}_M & \mathbf{1}_M & 0 \\ \mathbf{1}_M & \mathbf{0}_M & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{1.32}$$

where $\epsilon = +1$ for $SO(2M)$, while $\epsilon = -1$ for $USp(2M)$; the second matrix is for $SO(2M+1)$.

The Lagrangian density reads

$$\mathcal{L} = \mathrm{Tr}\Big[-\frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2g^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \mathcal{D}_\mu H \left(\mathcal{D}^\mu H\right)^\dagger \\ - \frac{e^2}{4} \left| X^0 t^0 - \frac{\xi}{N} \mathbf{1}_N \right|^2 - \frac{g^2}{4} \left| X^a t^a \right|^2 \Big] , \tag{1.33}$$

with the field strength, gauge fields and covariant derivative denoted as

$$F_{\mu\nu} = F^0_{\mu\nu} t^0 \, , F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu \, , \tag{1.34}$$

$$\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \left[A_\mu, A_\nu \right] \, , \quad A_\mu = A^a_\mu t^a \, , \tag{1.35}$$

$$\mathcal{D}_\mu = \partial_\mu + i A^0_\mu t^0 + i A^a_\mu t^a \, . \tag{1.36}$$

A^0_μ is the gauge field associated with $U(1)$ and A^a_μ are the gauge fields of G'. The matter scalar fields are written as an $N \times N_{\mathrm{F}}$ complex color (vertical) – flavor (horizontal) mixed matrix H. It can be expanded as

$$X = H H^\dagger = X^0 t^0 + X^a t^a + X^\alpha t^\alpha \, , \tag{1.37}$$

$$X^0 = 2\,\mathrm{Tr}\left(H H^\dagger t^0 \right) \, , \tag{1.38}$$

$$X^a = 2\,\mathrm{Tr}\left(H H^\dagger t^a \right) \, . \tag{1.39}$$

e and g are the $U(1)$ and G' coupling constants, respectively, while ξ is a real constant. t^0 and t^a stand for the $U(1)$ and G' generators, respectively, and finally, $t^\alpha \in \mathfrak{g}'_\perp$, where $\mathfrak{g}'_\perp$ is the orthogonal complement of the Lie algebra $\mathfrak{g}'$ in $\mathfrak{su}(N)$.

We have derived this theory from our $\mathcal{N} = 2$ theory and thus this theory is automatically at critical coupling – i.e. it is in the BPS limit. In order to keep the system in the Higgs phase, we take $\xi > 0$. The model has a gauge symmetry acting from the left on H and a flavor symmetry acting from the right. First we note that this theory has a continuous Higgs vacuum which was discussed in detail in Ref. [5]. In this Thesis we choose mainly to work in a particular point of the vacuum manifold

$$\langle H \rangle = \sqrt{\frac{\xi}{N}} \mathbf{1}_N \, , \tag{1.40}$$

namely, in the maximally "color-flavor-locked" Higgs phase of the theory. We have set $N_{\rm F} = N$ which is the minimal number of flavors allowing for such a vacuum. The existence of a continuous vacuum degeneracy implies the emergence of vortices of semi-local type as we shall see later.

The static tension, assuming independence of x^3 can be written as

$$T = \mathrm{Tr} \int_{\mathbb{C}} \left[\frac{1}{e^2} F_{12}^2 + \frac{1}{g^2} \hat{F}_{12}^2 + \mathcal{D}_i H \left(\mathcal{D}_i H\right)^\dagger + \frac{e^2}{4} \left| X^0 t^0 - \frac{\xi}{N} \mathbf{1}_N \right|^2 + \frac{g^2}{4} \left| X^a t^a \right|^2 \right] . \tag{1.41}$$

Using the following useful identity

$$\begin{aligned} \mathcal{D}_i H \left(\mathcal{D}_i H\right)^\dagger = \left(D_1 \pm i\mathcal{D}_2\right) H \left[\left(\mathcal{D}_1 \pm i\mathcal{D}_2\right) H\right]^\dagger \mp \left(F_{12} + \hat{F}_{12}\right) H H^\dagger \\ \mp i\epsilon^{ij} \partial_i \left(\left(\mathcal{D}_j H\right) H^\dagger\right) , \end{aligned} \tag{1.42}$$

and choosing the upper sign (by convention – which corresponds to studying vortices instead of anti-vortices), we can perform the Bogomol'nyi completion and obtain the tension as

$$\begin{aligned} T = \int_{\mathbb{C}} \mathrm{Tr} \Bigg[& \frac{1}{e^2} \left| F_{12} - \frac{e^2}{2} \left(X^0 t^0 - \frac{\xi}{N} \mathbf{1}_N \right) \right|^2 + \frac{1}{g^2} \left| \hat{F}_{12} - \frac{g^2}{2} X^a t^a \right|^2 + 4 \left| \bar{\mathcal{D}} H \right|^2 \\ & - \frac{\xi}{N} F_{12} - i\epsilon^{ij} \partial_i \left(\left(\mathcal{D}_j H\right) H^\dagger \right) \Bigg] \end{aligned} \tag{1.43}$$

$$\geq - \frac{\xi}{\sqrt{2N}} \int_{\mathbb{C}} F_{12}^0 \geq 0 , \tag{1.44}$$

where $2\bar{\mathcal{D}} \equiv \mathcal{D}_1 + i\mathcal{D}_2$ is used along with the standard complex coordinates $z = x^1 + ix^2$ and all fields are taken to be independent of x^3. When the inequality is saturated (BPS condition), the tension is simply

$$T = 2\pi\xi\nu, \qquad \nu = -\frac{1}{2\pi\sqrt{2N}} \int_{\mathbb{C}} F_{12}^0 , \tag{1.45}$$

where ν is the $U(1)$ winding number of the vortex. This leads immediately to the BPS equations for the vortex

$$\bar{\mathcal{D}} H = \bar{\partial} H + i\bar{A} H = 0 , \tag{1.46}$$

$$F_{12}^0 = e^2 \left(\mathrm{Tr}\left(H H^\dagger t^0\right) - \frac{\xi}{\sqrt{2N}} \right) , \tag{1.47}$$

$$F_{12}^a = g^2\, \mathrm{Tr}\left(H H^\dagger t^a\right) . \tag{1.48}$$

The corresponding tension density for these solutions can be rewritten from the tension functional (1.43) by using the BPS equation (1.46) as

$$\mathcal{E} = -\xi F_{12}^0 + \frac{1}{2} \partial_i^2 \mathrm{Tr}\left(H H^\dagger\right) , \tag{1.49}$$

which in turn can be rewritten by means of the second BPS equation (1.47) as

$$\mathcal{E} = -\xi F_{12}^0 + \frac{1}{e^2}\sqrt{\frac{N}{2}}\partial_i^2 F_{12}^0 \ . \tag{1.50}$$

The first BPS equation (1.46) can be solved [29, 100, 136] by the Ansatz

$$H = S^{-1}(z,\bar{z})H_0(z) \ , \qquad \bar{A} = -iS^{-1}(z,\bar{z})\bar{\partial}S(z,\bar{z}) \ , \tag{1.51}$$

where S belongs to the complexification of the gauge group, $S \in \mathbb{C}^* \times G'^{\mathbb{C}}$. $H_0(z)$, holomorphic in z, is called the *moduli matrix* [98, 99], which contains all moduli parameters of the vortices as will be seen below.

A gauge invariant object can be constructed as $\Omega = SS^\dagger$. It will, however, prove convenient to split this into the $U(1)$ part and the G' part, such that $S = s\,S'$ and analogously $\Omega = \omega\,\Omega'$, $\omega = |s|^2$, $\Omega' = S'S'^\dagger$. In terms of ω the tension (1.45) can be rewritten as

$$T = 2\pi\xi\nu = 2\xi\int_{\mathbb{C}}\partial\bar{\partial}\log\omega \ , \qquad \nu = \frac{1}{\pi}\int_{\mathbb{C}}\partial\bar{\partial}\log\omega \ , \tag{1.52}$$

and ν determines the asymptotic behavior of the Abelian field as

$$\omega = ss^\dagger \sim |z|^{2\nu} \ , \qquad \text{for } |z| \to \infty \ . \tag{1.53}$$

The moduli matrix $H_0(z)$ is defined up to equivalence relations of the form

$$\{H_0, S\} \sim V_e\, V'(z)\,\{H_0, S\} \ , \tag{1.54}$$

with the V-transformation taking part of the algebra $\mathfrak{g}'$

$$V'(z)^{\mathrm{T}} J V'(z) = J \ . \tag{1.55}$$

We define $\Omega_0 \equiv H_0 H_0^\dagger$ and obtain the following system of partial differential equations by inserting the Ansatz (1.51) into the remaining BPS equations (1.47) and (1.48)

$$\bar{\partial}\partial\log\omega = -\frac{e^2}{4N}\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right) - \xi\right) \ , \tag{1.56}$$

$$\bar{\partial}\left(\Omega'\partial\Omega'^{-1}\right) = \frac{g^2}{4\omega}\left(\Omega_0\Omega'^{-1} - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right) \ , \tag{1.57}$$

which are the master equations in the case of $G' = SU(N)$, while in the case of $G' = SO(N)$ or $G' = USp(N = 2M)$ we have

$$\bar{\partial}\partial\log\omega = -\frac{e^2}{4N}\left(\mathrm{Tr}\left(\frac{1}{\omega}\Omega_0\Omega'^{-1}\right) - \xi\right) \ , \tag{1.58}$$

$$\bar{\partial}\left(\Omega'\partial\Omega'^{-1}\right) = \frac{g^2}{8\omega}\left(\Omega_0\Omega'^{-1} - J^\dagger\left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}} J\right) \ . \tag{1.59}$$

The boundary conditions at $|z| \to \infty$ are

$$\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0 \Omega'^{-1}\right) = \xi\,, \quad \Omega_0 \Omega'^{-1} = J^\dagger \left(\Omega_0 \Omega'^{-1}\right)^{\mathrm{T}} J\,. \tag{1.60}$$

We assume the existence and uniqueness for the solutions to these equations. There are at least two justifications for this. One is the fact that in the strong coupling limit ($e, g \to \infty$) these can be algebraically and uniquely solved, which we will pursue in depth in Chap. 8. The other relies on the index theorem: the number of the moduli parameters encoded in H_0 coincides with that obtained from the index theorem [3] as we will show in Chap. 7.

The vortex solutions are characterized by a rational number $\nu > 0$, being the $U(1)$ winding number. ν will be found to be quantized in half-integers ($\nu = k/2$) for the groups $G' = SO(2M), USp(2M)$ with $k \in \mathbb{Z}_+$; $\nu = k$ (integers) for $G' = SO(2M+1)$; finally $\nu = k/N$ for $G' = SU(N)$, as well known. The integer k denotes the vortex number: $k = 1$ corresponds to the minimal vortex in all cases.

The key idea which enables us to extend the moduli-matrix formalism to general gauge groups, is to consider the holomorphic invariants $I^i_{G'}(H)$ made of H, which are invariant under $G'^{\mathbb{C}}$, with i labeling them. If the $U(1)$ charge of the i-th invariant $I^i_{G'}(H)$ is n_i, the following relation

$$I^i_{G'}(H) = I^i_{G'}\left(s^{-1} S'^{-1} H_0\right) = s^{-n_i} I^i_{G'}\left(H_0(z)\right)\,, \tag{1.61}$$

holds. If the boundary condition is given by

$$I^i_{G'}(H)\Big|_{|z|\to\infty} = I^i_{\mathrm{vev}}\, e^{i\nu n_i \theta}\,, \tag{1.62}$$

where $\nu\, n_i$ is the number of zeros of $I^i_{G'}$, it follows that

$$I^i_{G'}(H_0) = s^{n_i} I^i_{G'}(H) \sim I^i_{\mathrm{vev}}\, z^{\nu\, n_i}\,, \quad |z| \to \infty\,. \tag{1.63}$$

As $I^i_{G'}(H_0(z))$ are holomorphic, the above condition implies that $I^i_{G'}(H_0(z))$ are *polynomials* in z. We find that $\nu\, n_i$ must be a positive integer for all i:

$$\nu\, n_i \in \mathbb{Z}_+ \quad \Rightarrow \quad \nu = \frac{k}{n_0}\,, \quad k \in \mathbb{Z}_+\,, \tag{1.64}$$

where (gcd denotes the greatest common divisor)

$$n_0 \equiv \mathrm{gcd}\left\{n_i \mid I^i_{\mathrm{vev}} \neq 0\right\}\,. \tag{1.65}$$

Note that a $U(1)$ gauge transformation $e^{\frac{2\pi i}{n_0}}$ leaves invariant $I^i_{G'}(H)$:

$$I^i_{G'}(H') = e^{\frac{2\pi i n_i}{n_0}} I^i_{G'}(H) = I^i_{G'}(H)\,, \tag{1.66}$$

i.e. the phase rotation $e^{\frac{2\pi i}{n_0}} \in \mathbb{Z}_{n_0}$ changes no physics, and the true gauge group is thus

$$G = \frac{U(1) \times G'}{\mathbb{Z}_{n_0}}\,. \tag{1.67}$$

where $\mathbb{Z}_{n_0}$ is the center of G'. A simple homotopy argument tells us that $1/n_0$ is the $U(1)$ winding for the minimal ($k = 1$) vortex configuration. Finally, for a given k the following important relation holds

$$I^i_{G'}(H_0) = I^i_{\rm vev} z^{\frac{kn_i}{n_0}} + \mathcal{O}\left(z^{\frac{kn_i}{n_0}-1}\right) , \tag{1.68}$$

which implies non-trivial constraints on $H_0(z)$.

The explicit form of the constraints follows from this general discussion. For $G' = SU(N)$ (with N flavors), there exists only one invariant

$$I_{SU} = \det(H) , \tag{1.69}$$

with charge N. Thus the minimal winding ($1/n_0$) is equal to $1/N$ and the condition for k vortices is given by:

$$A_{N-1}: \quad \det H_0(z) = z^k + \mathcal{O}\left(z^{k-1}\right), \quad \nu = \frac{k}{N} . \tag{1.70}$$

For $G' = SO(N), USp(2M)$, there are $N(N \pm 1)/2$ invariants

$$(I_{SO,USp})^r{}_s = (H^{\rm T} J H)^r{}_s , \quad 1 \leq r \leq s \leq N , \tag{1.71}$$

in addition to Eq. (1.69). The constraints are:

$$\begin{aligned} C_M, D_M: \quad & H_0^{\rm T}(z) J H_0(z) = z^k J + \mathcal{O}\left(z^{k-1}\right) , \quad \nu = \frac{k}{2} , \\ B_M: \quad & H_0^{\rm T}(z) J H_0(z) = z^{2k} J + \mathcal{O}\left(z^{2k-1}\right) , \quad \nu = k , \end{aligned} \tag{1.72}$$

for $G' = SO(2M), USp(2M)$ and $SO(2M + 1)$, respectively. As anticipated, vortices in the $SO(2M + 1)$ model have integer $U(1)$ windings [118].

Explicitly, the minimal vortices for $G' = SU(N)$ is given by the following moduli matrix

$$H_0(z) = \begin{pmatrix} z - a & 0 \\ \vec{b}^{\rm T} & \mathbf{1}_{N-1} \end{pmatrix} , \tag{1.73}$$

while in the case of $G' = SO(2M)$ or $G' = USp(2M)$ theories are given respectively by the moduli matrices

$$H_0(z) = \begin{pmatrix} z\mathbf{1}_M - \mathbf{A} & \mathbf{C}_{S/A} \\ \mathbf{B}_{A/S} & \mathbf{1}_M \end{pmatrix} . \tag{1.74}$$

The moduli parameters are all complex. For $SU(N)$, a is just a number; $\vec{b}^{\rm T}$ is a column vector. For $SO(2M)$ or $USp(2M)$, the matrix $\mathbf{C}_{S/A}$ for instance is symmetric or antisymmetric, respectively. And vice versa for $\mathbf{B}$.

The index theorem gives the complex dimension of the moduli space

$$\dim_{\mathbb{C}}\left(\mathcal{M}_{G',k}\right) = \frac{kNN_{\rm F}}{n_0} . \tag{1.75}$$

This was obtained in Ref. [9] for $SU(N)$; a general proof in the case of generic gauge groups is given in Chap. 7.

Except for the $SU(N)$ case, our model has a non-trivial Higgs branch (flat directions). The color-flavor locked vacuum $\langle H\rangle \propto \mathbf{1}_N$ is just one of the possible (albeit the most symmetric) choices for the vacuum; our discussion can readily be generalized to a generic vacuum on the Higgs branch. This fact, however, implies that our non-Abelian vortices have "semi-local" moduli (see Achucarro et. al. [85, 15]), even for $N_{\rm F} = N$.

1.2.1 Local vortices

For various considerations, we are interested in knowing which of the moduli parameters describe the so-called local vortices, the ANO-type vortices with exponential tails. To identify these, let us first consider generic points in the moduli space. In the strong coupling limit, our theory reduces to a NLσM, with the (classical) vacuum moduli $\mathcal{M}_{\rm vac}$ as its target space. In such a limit, semi-local vortices with non-zero size moduli reduce to the so-called σ model lumps. The local vortices on the other hand shrink to singular configurations. It is well-known that lumps are characterized by $\pi_2(\mathcal{M}_{\rm vac})$ with a wrapping around a 2-cycle inside $\mathcal{M}_{\rm vac}$. Even at finite gauge coupling, asymptotic configurations of semi-local vortices can be well approximated by lumps.

Now the moduli space of vacua $\mathcal{M}_{\rm vac}$ in supersymmetric models is parametrized by holomorphic invariants $I^I_G(H)$ $(I = 1, 2, \ldots)$ of the complexified gauge group $G^{\mathbb{C}}$ [157]. In our case, $G = G' \times U(1)$, with the common $U(1)$ charge of the scalar fields H, all the $G^{\mathbb{C}}$ invariants $I^I_G(H)$ can be written using the $G'^{\mathbb{C}}$ invariants $I^i_{G'}(H)$. For instance from $I^i_{G'}$ and $I^j_{G'}$ with $n_i = n_j$, one can construct

$$I_G^{(i,j)}(H) \equiv \frac{I^i_{G'}(H)}{I^j_{G'}(H)} = \frac{I^i_{G'}(H_0(z))}{I^j_{G'}(H_0(z))}\,, \tag{1.76}$$

where use was made of Eq. (1.61). The last line defines, so-called, (generalized) rational maps. This observation allows us to define local vortices. While the asymptotic region of *semi-local* vortices is mapped to some domain of $\mathcal{M}_{\rm vac}$, that around the *local* vortices is mapped into a single point. Therefore, all the $G^{\mathbb{C}}$ invariants $I^I_G(H)$ must be constant for the latter. All the $I_{G'}(H)$'s have zeros at the vortex positions and winding around them as seen in Eq. (1.62). These facts, together with Eq. (1.76), imply that all $I^i_{G'}(H_0(z))$'s with the same n_i must have common zeros

$$I^i_{G'}\left(H_{0,\rm local}\right) = \left[\prod_{\ell=1}^{k}(z - z_{0\ell})\right]^{\frac{n_i}{n_0}} I^i_{\rm vev}\,. \tag{1.77}$$

For $G' = SO(2M), USp(2M)$ with $I_{SO,USp}$ of Eq. (1.71) we find that the condition for vortices to be of local type is

$$H^{\rm T}_{0,\rm local}(z) J H_{0,\rm local}(z) = \prod_{\ell=1}^{k}(z - z_{0\ell})\ J\,. \tag{1.78}$$

Let us now discuss a few concrete examples. The general solution for the minimal vortex (1.74) with $G' = SU(N)$ is reduced to a local vortex if we restrict it to be of the form

$$H_{0,\text{local}}(z) = \begin{pmatrix} z-a & 0 \\ \vec{b}^{\,\mathrm{T}} & \mathbf{1}_{N-1} \end{pmatrix} , \tag{1.79}$$

while for $G' = \{SO(2M), USp(2M)\}$ we have

$$H_{0,\text{local}}(z) = \begin{pmatrix} (z-a)\,\mathbf{1}_M & 0 \\ \mathbf{B}_{A/S} & \mathbf{1}_M \end{pmatrix} . \tag{1.80}$$

The vortex position is given by a. $\vec{b}^{\,\mathrm{T}}$ in the case of $SU(N)$ and $\mathbf{B}_{A/S}$ in the case of $SO(2M)$ or $USp(2M)$ encode the Nambu-Goldstone modes associated with the breaking of the color-flavor symmetry by the vortex $G'_{\mathrm{C+F}} \to K_{G'}$. The moduli spaces are direct products of a complex number and the Hermitian symmetric spaces

$$\mathcal{M}^{\text{local}}_{G',k=1} \simeq \mathbb{C} \times \frac{G'_{\mathrm{C+F}}}{K_{G'}} , \tag{1.81}$$

$K_{SU(N)} = SU(N-1) \times U(1)$ while $K_{SO(2M),USp(2M)} = U(M)$. The moduli space in the case of $G' = SU(N)$ thus reads

$$\mathcal{M}^{\text{local}}_{SU(N),k=1} = \mathbb{C} \times \frac{SU(N)}{SU(N-1)\times U(1)} \simeq \mathbb{C} \times \mathbb{C}P^{N-1} , \tag{1.82}$$

while in the case of $G' = SO(2M)$ and $G' = USp(2M)$ we have

$$\mathcal{M}^{\text{local}}_{SO(2M),k=1} = \mathbb{C} \times \frac{SO(2M)}{U(M)} , \qquad \mathcal{M}^{\text{local}}_{USp(2M),k=1} = \mathbb{C} \times \frac{USp(2M)}{U(M)} . \tag{1.83}$$

These results for $SU(N)$ and $SO(2M)$ are well-known [9, 97, 118].

The matrices (1.80) describe just one patch of the moduli space. In order to define the manifold globally we need a sufficient number of patches. The number of patches is N for $G' = SU(N)$ and 2^M for $G' = SO(2M), USp(2M)$. The transition functions correspond to the V-equivalence relations [100, 111, 136]. In the case of $G' = SO(2M)$, the patches are given by permutation of the i-th and the $(M+i)$-th columns in (1.80). We find that no regular transition functions connect the odd and even permutations (patches), hence the moduli space consists of two disconnected copies of $SO(2M)/U(M)$ [118]. The complex dimensions of the moduli spaces are

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{local}}_{SO(2M),k=1} = \frac{1}{2}M(M-1) + 1 , \tag{1.84}$$

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{local}}_{USp(2M),k=1} = \frac{1}{2}M(M+1) + 1 . \tag{1.85}$$

G'	A_{N-1}	B_M	C_M, D_M	E_6	E_7	E_8	F_4	G_2
R	N	$2M+1$	$2M$	27	56	248	26	7
rank inv	$-$	2	2	3	$2,4$	$2,3,8$	$2,3$	$2,3$
n_0	N	1	2	3	2	1	1	1

Table 1.1: The dimension of the fundamental representation (R), the rank of the other invariants [10] and the minimal tension $\nu = 1/n_0$ i.e. the center $\mathbb{Z}_{n_0}$ of G'. The determinant of the $R \times R$ matrix gives one invariant with charge, $\dim R$.

1.2.2 Exceptional groups

$\boldsymbol{E}_6$: There is a rank-3 symmetric tensor: Γ_{ijk}. The conditions on the moduli matrix are

$$\Gamma_{i_1 i_2 i_3}(H_0)^{i_1}{}_{j_1}(H_0)^{i_2}{}_{j_2}(H_0)^{i_3}{}_{j_3} \sim \Gamma_{j_1 j_2 j_3} z^k , \tag{1.86}$$

and the $U(1)$ winding number is quantized as $\nu = k/3$.

$\boldsymbol{E}_7$: There are 2 invariant tensors: d_{ijkl} and f_{ij} respectively of rank 4 and 2. The moduli matrix is constrained as:

$$\begin{aligned} d_{i_1 i_2 i_3 i_4}(H_0)^{i_1}{}_{j_1}(H_0)^{i_2}{}_{j_2}(H_0)^{i_3}{}_{j_3}(H_0)^{i_4}{}_{j_4} &\sim d_{j_1 j_2 j_3 j_4} z^{2k} , \\ f_{i_1 i_2}(H_0)^{i_1}{}_{j_1}(H_0)^{i_2}{}_{j_2} &\sim f_{j_1 j_2} z^k, \end{aligned} \tag{1.87}$$

and the vortices are quantized in half integers: $\nu = k/2$.

$\boldsymbol{G}_2,\ \boldsymbol{F}_4,\ \boldsymbol{E}_8$: See Table 1.1 for the list of the invariant tensors and the winding numbers.

1.2.3 Strong coupling limit

If we now consider taking the strong gauge coupling limit, the master equations in the case of $G' = SU(N)$, Eqs. (1.56) and (1.57) reduce to the algebraic matrix equations

$$0 = \frac{1}{\omega}\mathrm{Tr}\left(\Omega_0 \Omega'^{-1}\right) - \xi , \tag{1.88}$$

$$0 = \Omega_0 \Omega'^{-1} - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\Omega_0 \Omega'^{-1}\right) , \tag{1.89}$$

which have the solution (that also coincides with the boundary conditions)

$$\omega_\infty = \frac{N}{\xi}\left(\det \Omega_0\right)^{\frac{1}{N}} , \quad \Omega'_\infty = \left(\det \Omega_0\right)^{-\frac{1}{N}} \Omega_0 , \quad \Omega_\infty = \omega_\infty \Omega'_\infty , \tag{1.90}$$

while in the case $G' = SO(N)$ or $G' = USp(2M)$ we have

$$0 = \frac{1}{\omega}\mathrm{Tr}\left(\Omega_0 \Omega'^{-1}\right) - \xi , \tag{1.91}$$

$$0 = \Omega_0 \Omega'^{-1} - J^\dagger \left(\Omega_0 \Omega'^{-1}\right)^{\mathrm{T}} J , \tag{1.92}$$

which in turn have the solution

$$\omega_\infty = \frac{1}{\xi}\mathrm{Tr}\sqrt{M^\dagger M} \; ,$$
$$\Omega'_\infty = H_0(z)\frac{\mathbf{1}_N}{\sqrt{M^\dagger M}}H_0^\dagger(z) \; , \quad \Omega_\infty = \omega_\infty \Omega'_\infty \; , \tag{1.93}$$

where

$$M \equiv H_0^{\mathrm{T}}(z) J H_0(z) \; , \tag{1.94}$$

is the meson field.

This limit corresponds to considering the low-energy effective theory where the mass of the gauge bosons is infinitely heavy and it can be physically justified to integrate them out, for instance in the path integral formalism. In this limit, the gauge theory becomes a NLσM and the vortices become so-called NLσM lumps or simply lumps.

A useful technology to find exactly these low-energy effective solutions is the Kähler quotient or for $\mathcal{N}=2$ supersymmetric theories, the hyper-Kähler quotient. The underlying reason for this is the fact that the Higgs branch of $\mathcal{N}=2$ ($\mathcal{N}=1$) supersymmetric QCD is hyper-Kähler (Kähler). We will briefly introduce this formalism in the next Section.

1.3 Kähler quotient

Let us first give a brief review on the $SU(N)$ Kähler quotient. We start with the $\mathcal{N}=1$, $SU(N)$ supersymmetric Yang-Mills theory with N_{F} chiral superfields Q (i.e. an N-by-N_{F} matrix) in the fundamental representation of $SU(N)$. Denote the $SU(N)$ vector multiplet by a superfield V', then a Kähler potential for the system is

$$K_{SU(N)} = \mathrm{Tr}\left[QQ^\dagger e^{-V'}\right] \; . \tag{1.95}$$

The Lagrangian is invariant under the complexification of the gauge group,

$$SU(N)^{\mathbb{C}} = SL(N,\mathbb{C}) \; ,$$

given by

$$Q \to e^{i\Lambda'}Q \; , \quad e^{V'} \to e^{i\Lambda'}e^{V'}e^{-i\Lambda'^\dagger} \; , \qquad e^{i\Lambda'} \in SU(N)^{\mathbb{C}} \; . \tag{1.96}$$

while we will not consider any superpotentials here.

As well-known, the kinetic term of the vector supermultiplet

$$\frac{1}{4g^2}\int d^2\theta\, W^\alpha W_\alpha + \frac{1}{4g^2}\int d^2\bar{\theta}\, \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \; , \tag{1.97}$$

includes a so-called D-term potential in the Wess-Zumino gauge, in which $SU(N)^{\mathbb{C}}$ is fixed to $SU(N)$

$$V_D = \frac{g^2}{2}\left(D^A\right)^2 \; , \qquad D^A = \mathrm{Tr}_{\mathrm{F}}\left(Q_{\mathrm{wz}}^\dagger t^A Q_{\mathrm{wz}}\right) \; , \tag{1.98}$$

where t^A are $SU(N)$ generators and $Q_{\rm wz}$ denotes Q in the Wess-Zumino gauge. The vacuum condition $D^A = 0$ (D-flatness) allows both for an unbroken phase and for the Higgs phase. It implies that $Q_{\rm wz}Q_{\rm wz}^\dagger \propto \mathbf{1}_N$ holds in the vacuum. On the Higgs branch ($\operatorname{rank} Q_{\rm wz} = N$), the gauge fields acquire masses of the order $g\langle Q\rangle$ by the Higgs mechanism. If we restrict ourselves to energies much below the mass scale, we can omit the massive gauge fields. In order to get a low-energy effective theory, it will prove useful to consider a limit where the gauge coupling is taken to infinity: $g \to \infty$. In this limit, the vector multiplet becomes infinitely massive and looses the kinetic term. Thus, it reduces to merely an auxiliary field. At the same time the D-term potential forces $Q_{\rm wz}$ to take a value in the vacuum $D^A = 0$. Thus the low-energy effective theory is a NLσM, whose target space is the vacuum of the gauge theory

$$\mathcal{M}_{SU(N)} = \left\{Q_{\rm wz} \mid Q_{\rm wz}Q_{\rm wz}^\dagger \propto \mathbf{1}_N \,,\, \operatorname{rank} Q_{\rm wz} = N\right\} /SU(N)\,. \tag{1.99}$$

The real dimension of the manifold is

$$2NN_{\rm F} - (N^2-1) - (N^2-1) = 2N(N_{\rm F}-N) + 2\,.$$

Before fixing the complexified gauge symmetry $SU(N)^{\mathbb{C}}$, for instance by choosing the Wess-Zumino gauge as above, we can take the strong coupling limit immediately. This gives another description of the NLσM. The Lagrangian consists of only one term i.e. Eq. (1.95) and we do not have the D-term conditions anymore, however, instead we have the complex fields Q and the complexified gauge group $SU(N)^{\mathbb{C}}$. The target space is expressed by

$$\mathcal{M}_{SU(N)} = \{Q \,|\, \operatorname{rank} Q = N\} /\!\!/ SU(N)^{\mathbb{C}}\,. \tag{1.100}$$

In order for this quotient to be well-defined, the action of $SU(N)^{\mathbb{C}}$ must be free on Q. Namely, the gauge symmetry should be completely broken, thus we are going to study the *full* Higgs phase. The complex dimension of the manifold is

$$NN_{\rm F} - (N^2-1) = N(N_{\rm F}-N) + 1\,,$$

which coincides with the dimension of (1.99). The two expressions (1.99) and (1.100) of the target space are in fact identical. One can find a relation between them by solving the equations of motion for V'. It determines the traceless part as $QQ^\dagger e^{-V'} \propto \mathbf{1}_N$. Taking $\operatorname{Tr} V' = 0$ into account, V' is uniquely determined as

$$V' = \log QQ^\dagger - \frac{1}{N}\mathbf{1}_N \log\det(QQ^\dagger)\,, \tag{1.101}$$

if and only if $\operatorname{rank} Q$ is maximal, which corresponds to the full Higgs phase. Now we can find an explicit map from the quotient (1.100) to the vacuum configuration (1.99):

$$Q_{\rm wz} = e^{-V'/2}Q = \left[\det(QQ^\dagger)\right]^{\frac{1}{2N}} \frac{1}{\sqrt{QQ^\dagger}}Q\,. \tag{1.102}$$

There is still another way to express the same NLσM. As explained above, the target space is nothing but the classical moduli space of vacua of the original supersymmetric gauge theory. As discussed in Ref. [157] it can be described by holomorphic invariants of the complexified gauge group. Hence, the Kähler potential on the NLσM should be expressed in terms of such holomorphic invariants. The holomorphic invariants of $SU(N)^{\mathbb{C}}$ are the baryon operators

$$B^{\langle A_1\cdots A_N\rangle} \equiv \det Q^{\langle A_1\cdots A_N\rangle} = \epsilon^{i_1\cdots i_N} Q_{i_1}{}^{A_1}\cdots Q_{i_N}{}^{A_N}\,, \tag{1.103}$$

where $Q^{\langle A_1\cdots A_N\rangle}$ denotes an N-by-N minor matrix of the N-by-$N_{\rm F}$ matrix Q as $(Q^{\langle A\rangle})_i^j = Q_i{}^{A_j}$. We often abbreviate the label $\langle A_1\cdots A_N\rangle$ as $\langle A\rangle$. The important point is that all the $B^{\langle A\rangle}$'s are not independent and they satisfy the so-called Plücker relations

$$B^{\langle A_1\cdots A_{N-1}[B_1\rangle}B^{\langle B_2\cdots B_{N+1}]\rangle} = 0\,. \tag{1.104}$$

Furthermore, the condition for having the full Higgs phase requires that at least one of the $B^{\langle A\rangle}$'s must take a non-zero value. Actually, we can reconstruct Q modulo $SU(N)$ gauge symmetry by solving the Plücker relations with one non-zero $B^{\langle A\rangle}$ as the starting point. That is, the holomorphic invariants with the Plücker relations give us the same information as the two descriptions above. Hence, the target space is also expressed as

$$\mathcal{M}_{SU(N)} = \left\{B^{\langle A\rangle} \mid \text{Eq. (1.104)}\right\} - \left\{B^{\langle A\rangle} = 0\,,\forall\langle A\rangle\right\}. \tag{1.105}$$

Let us show the metric on the target space. It can be derived from the Kähler potential (1.95) and is represented by

$$K_{SU(N)} = N\left[\det(QQ^\dagger)\right]^{\frac{1}{N}} = N\left(\sum_{\langle A\rangle}\left|B^{\langle A\rangle}\right|^2\right)^{\frac{1}{N}}\,. \tag{1.106}$$

The appearance of the Nth root reflects the fact that the $U(1)$ charge of the invariants is N, as we will see soon. Notice that the (partial) Coulomb phase ($\det(QQ^\dagger) = 0$) shrinks to a point of the target manifold from the point of view of the NLσM and a trace of this fact is seen as the $\mathbb{Z}_N$ conifold singularity at that point. In the simple case with $N_{\rm F} = N$, one can find the NLσM on an orbifold $\mathbb{C}/\mathbb{Z}_N$. At the singularity, the vector multiplet becomes massless and the gauge symmetry is restored. We have to take all the massless fields into account there, namely we cannot restrict ourselves to the NLσM, but we have to return to the original gauge theory.

This singularity (that is, the Coulomb phase) is removed once the overall $U(1)$ phase is gauged and the so-called Fayet-Iliopoulos (FI) parameter ξ (> 0) [158] is introduced for that $U(1)$. Let us consider a $U(1)\times SU(N)$ gauge theory. Still we neglect the kinetic terms associated with the vector multiplet, such that the vector multiplet is an auxiliary superfield. The Kähler potential is given by

$$K_{U(1)\times SU(N)} = \mathrm{Tr}\left[QQ^\dagger e^{-V_e}e^{-V'}\right] + \xi V_e = e^{-V_e}K_{SU(N)} + \xi V_e\,, \tag{1.107}$$

where V_e is a $U(1)$ vector supermultiplet and the chiral fields Q have $U(1)$ charge $+1$. The D-flatness condition for the overall $U(1)$ implies that

$$Q_{\rm wz}Q_{\rm wz}^\dagger = \frac{\xi}{N}\mathbf{1}_N\,. \tag{1.108}$$

The target space of the NLσM becomes a compact space; the complex Grassmannian manifold[2]

$$\mathcal{M}_{U(1)\times SU(N)} = Gr_{N_{\rm F},N} \simeq SU(N_{\rm F})/[SU(N_{\rm F}-N)\times SU(N)\times U(1)]\,. \qquad (1.109)$$

As in the case above, we have three different representations

$$\begin{aligned}\mathcal{M}_{U(1)\times SU(N)} &= \left\{Q_{\rm wz}\,\middle|\,Q_{\rm wz}Q_{\rm wz}^\dagger = \frac{\xi}{N}\mathbf{1}_N\right\}\Big/\,(U(1)\times SU(N))\\ &= \{Q\,|\,{\rm rank}\,Q = N\}\,/\!\!/(U(1)\times SU(N))^{\mathbb{C}}\\ &= \left(\left\{B^{\langle A\rangle}\mid \text{Eq. (1.104)}\right\}-\left\{B^{\langle A\rangle}=0\,,\forall\langle A\rangle\right\}\right)/\!\!/U(1)^{\mathbb{C}}\,. \end{aligned}\qquad (1.110)$$

A relation between $Q_{\rm wz}$ and Q is also found here by solving the equations of motion with respect to V' and V_e. The solution for V' is the same as Eq. (1.101) and the $U(1)$ part is then written as

$$V_e = \log\left(\xi^{-1}K_{SU(N)}\right)\,. \qquad (1.111)$$

Then the map from the quotient space to the vacuum configuration is given by

$$Q_{\rm wz} = e^{-V'/2-V_e/2}Q = \sqrt{\frac{\xi}{N}}\frac{1}{\sqrt{QQ^\dagger}}Q\,. \qquad (1.112)$$

The third expression in Eq. (1.110) shows the Plücker embedding of the Grassmannian space into a bigger space, the complex projective space $\mathbb{C}P^n$ with $n = \frac{N_{\rm F}!}{N!(N_{\rm F}-N)!}-1$. The Kähler potential can now be expressed by

$$K_{U(1)\times SU(N)} = \frac{\xi}{N}\log\det\left(QQ^\dagger\right) = \frac{\xi}{N}\log\left(\sum_{\langle A\rangle}\left|B^{\langle A\rangle}\right|^2\right)\,. \qquad (1.113)$$

The $1/N$ factor in front is the (inverse) $U(1)$ charge of the invariant $B^{\langle A\rangle}$. The FI parameter plays an important role: it forces the gauge symmetry $U(1)\times SU(N)$ to be fully broken, namely it hides the singularity at the origin, where the gauge symmetry is recovered.

The Grassmannian manifold is one of the Hermitian symmetric spaces. NLσMs on all Hermitian symmetric spaces can be obtained by imposing proper holomorphic constraints from F-terms, by which Hermitian symmetric spaces are embedded into $\mathbb{C}P^{N_{\rm F}-1}$ or the Grassmannian manifold [160].

1.3.1 Hyper-Kähler quotient

One can easily extend the above Kähler quotient to the hyper-Kähler quotient by considering a natural $\mathcal{N}=2$ supersymmetric extension. Here we will review the $U(1)\times SU(N)$ case only. The Kähler potential and the superpotential are given by

$$\tilde{K}_{U(1)\times SU(N)} = {\rm Tr}\left[QQ^\dagger e^{-V_e}e^{-V'} + \tilde{Q}^\dagger\tilde{Q}e^{V_e}e^{V'}\right] + \xi V_e\,, \qquad (1.114)$$

$$W = {\rm Tr}\left[Q\tilde{Q}\Sigma\right]\,, \qquad (1.115)$$

[2] The $U(N)$ Kähler quotient construction of the Grassmann manifold was first found in Ref. [159] in the superfield formalism.

respectively, where we have introduced $N_{\rm F}$ hypermultiplets $(Q,\tilde{Q}^\dagger)$ in the fundamental representation of $U(N)\simeq U(1)\times SU(N)$ and $U(N)$ vector superfields $(V,\Sigma)=(V'+V_e\mathbf{1}_N,\Sigma)$. The complexified gauge transformation is given by

$$Q\to e^{i\Lambda}Q\ ,\quad \tilde{Q}\to\tilde{Q}e^{-i\Lambda}\ ,\quad e^V\to e^{i\Lambda}e^Ve^{-i\Lambda^\dagger}\ ,\quad \Sigma\to e^{i\Lambda}\Sigma e^{-i\Lambda}\ , \tag{1.116}$$

where $\Lambda\in GL(N,\mathbb{C})$. The target space of the corresponding NLσM is a hyper-Kähler manifold, namely the cotangent bundle $T^\star Gr_{N_{\rm F},N}$ over the complex Grassmannian manifold $Gr_{N_{\rm F},N}$, endowed with the Lindström-Roček metric [56]. Let us obtain the Kähler potential with respect to $Q,\tilde{Q}$ without choosing the Wess-Zumino gauge. The equations of motion for Σ and V are

$$Q\tilde{Q}=0\ , \tag{1.117}$$

$$-QQ^\dagger e^{-V}+e^V\tilde{Q}^\dagger\tilde{Q}+\frac{\xi}{N}\mathbf{1}_N=0\ . \tag{1.118}$$

The first equation implies that $\tilde{Q}$ is orthogonal to Q. The rank of Q must be N due to the positive FI parameter ξ, while $\tilde{Q}$ can be zero. Therefore Q $(\tilde{Q}=0)$ parametrizes the base space $Gr_{N_{\rm F},N}$ with the total space being the cotangent bundle over it. Let us count the complex dimensions of the target space:

$$NN_{\rm F}+N_{\rm F}N-N^2-N^2=2N(N_{\rm F}-N)\ ,$$

where the first subtraction is the $U(N)^{\mathbb{C}}$ quotient and the second is the number of conditions given in Eq. (1.117). In order to solve the second matrix equation, we first multiply by $\sqrt{QQ^\dagger}e^{-V}$ from the left and by $\sqrt{QQ^\dagger}$ from the right,[3] such that the matrix equation becomes Hermitian

$$X^2-\frac{\xi}{N}X-\sqrt{QQ^\dagger}\tilde{Q}^\dagger\tilde{Q}\sqrt{QQ^\dagger}=0\ ,\qquad X\equiv\sqrt{QQ^\dagger}e^{-V}\sqrt{QQ^\dagger}\ . \tag{1.119}$$

Therefore, using $\det QQ^\dagger\neq 0$, we find the solution

$$\begin{aligned}V&=-\log\left[\frac{1}{\sqrt{QQ^\dagger}}X\frac{1}{\sqrt{QQ^\dagger}}\right]\ ,\\ \text{with}\ \ X&=\frac{\xi}{2N}\mathbf{1}_N+\sqrt{\sqrt{QQ^\dagger}\tilde{Q}^\dagger\tilde{Q}\sqrt{QQ^\dagger}+\frac{\xi^2}{4N^2}\mathbf{1}_N}\ .\end{aligned} \tag{1.120}$$

We will now switch to another description i.e. using holomorphic invariants. We have the following invariants of the $SU(N)^{\mathbb{C}}$ gauge group

$$B^{\langle A\rangle}=\det Q^{\langle A\rangle}\ ,\quad M=\tilde{Q}Q\ ,\quad \left(\tilde{B}_{\langle A\rangle}=\det\tilde{Q}_{\langle A\rangle}\right)\ . \tag{1.121}$$

In addition to the Plücker relations for the $B^{\langle A\rangle}$'s, there are constraints on the mesonic invariant M

$$M_B{}^{[A_1}B^{\langle A_2\cdots A_{N+1}]\rangle}=0\ ,\quad B^{\langle A_1\cdots A_{N-1}A'\rangle}M_{A'}{}^B=0\ . \tag{1.122}$$

[3] Note that the square root and the logarithm are uniquely defined for positive (semi-)definite Hermitian matrices. This point might be missed (at least in this context) in the physics literature so far.

Furthermore, $B^{\langle A\rangle}$ (and $\tilde{B}_{\langle A\rangle}$) are only defined up to $U(1)^{\mathbb{C}}$ equivalence transformations. After reconstructing Q from (some) non-vanishing $B^{\langle A\rangle}$, we can reconstruct $\tilde{Q}$ from the first condition and find the constraint $Q\tilde{Q} = 0$ from the second. Therefore, these invariants and their constraints describe the same target space, $T^\star Gr_{N_{\rm F},N}$. Plugging back the solution (1.120) into the Kähler potential (1.114), we obtain the Kähler potential in terms of these invariants [56, 58]

$$\begin{aligned}\tilde{K}_{U(1)\times SU(N)} &= K_{U(1)\times SU(N)} \\ &+ \frac{\xi}{N}\,\mathrm{Tr}_{\rm F}\left[\sqrt{\mathbf{1}_{N_{\rm F}} + \frac{4N^2}{\xi^2}MM^\dagger} - \log\left(\mathbf{1}_{N_{\rm F}} + \sqrt{\mathbf{1}_{N_{\rm F}} + \frac{4N^2}{\xi^2}MM^\dagger}\right)\right] .\end{aligned} \tag{1.123}$$

We have used $A^\dagger A = MM^\dagger$ and the cyclic property of a trace, i.e. for $A = \sqrt{QQ^\dagger}\tilde{Q}^\dagger$

$$\mathrm{Tr}\left[f(AA^\dagger) - f(\mathbf{0}_N)\mathbf{1}_N\right] = \mathrm{Tr}\left[f(A^\dagger A) - f(\mathbf{0}_{N_{\rm F}})\mathbf{1}_{N_{\rm F}}\right] . \tag{1.124}$$

This relation can easily be proven by expanding the function f around $AA^\dagger = \mathbf{0}_N$. Recall that the logarithm and the square root of a positive (semi-)definite Hermitian matrix can be calculated by diagonalization and therefore the cyclic property works not only for polynomial functions but for any function $f(x)$.

The hyper-Kähler quotient construction of the cotangent bundle over the Grassmann manifold has been reviewed here. For $N = 1$, the $U(1)$ hyper-Kähler quotient reduces to the cotangent bundle over the complex projective space, $T^\star\mathbb{C}P^{N_{\rm F}-1}$ [53, 54, 55], endowed with the Calabi metric [59]. The explicit Kähler potentials of the cotangent bundles over the other Hermitian symmetric spaces have recently been obtained by a rather different method [161, 162, 163, 164, 165, 166]. It is an open question if these manifolds can be obtained as a certain hyper-Kähler quotient.

We will not repeat the derivation of the $SU(N)$ hyper-Kähler quotient here. Explicit expressions can be found in the literature, see for instance [58, 167, 168]. It gives the cotangent bundle over the $SU(N)$ Kähler quotient derived in the last Subsection.

1.4 Abelian vortices, BPS-ness and integrability

After a tour in the non-Abelian vortices, we will now simplify them to Abelian ones, just for completeness and because we will need them in some cases, for instance to create the wall vortices in the next Section.

Let us take a pedestrian route to find exactly the same equations as in the last Sections with a much more elaborate machinery and this way see how supersymmetry minimizes the tension and in turn cancels the inter-static forces between multi-vortices. Hence, we will take the Lagrangian (1.33) and keep only the Abelian part

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu H\left(\mathcal{D}^\mu H\right)^\dagger - \frac{\beta e^2}{4}\left(HH^\dagger - \xi\right)^2 , \tag{1.125}$$

where our squark field is now just a single complex scalar (as opposed to a matrix) and we have inserted a parameter β as a prefactor of the potential and[4]

$$\mathcal{D}_\mu = \partial_\mu + \frac{i}{\sqrt{2}} A_\mu \, . \tag{1.126}$$

This is the Abelian Higgs model which possesses the celebrated Abrikosov-Nielsen-Olesen vortices [11, 12]. The equations of motion are

$$\mathcal{D}_\mu \mathcal{D}^\mu H + \frac{\beta e^2}{2} \left(|H|^2 - \xi \right) H = 0 \, , \tag{1.127}$$

$$\frac{1}{e^2} \partial_\mu F^{\mu\nu} - \frac{i}{\sqrt{2}} \left(H^\dagger \mathcal{D}^\nu H - H \left(\mathcal{D}^\nu H \right)^\dagger \right) = 0 \, . \tag{1.128}$$

Writing the static tension and assuming independence of x^3 we have

$$T = \int_{\mathbb{C}} \left[\frac{1}{2e^2} F_{12}^2 + |\mathcal{D}_i H|^2 + \frac{\beta e^2}{4} \left(|H|^2 - \xi \right)^2 \right] \, , \tag{1.129}$$

where $i = 1, 2$ is summed over and we rewrite this tension using the following identity

$$|\mathcal{D}_i H|^2 = |(\mathcal{D}_1 \pm i\mathcal{D}_2) H|^2 \mp \frac{1}{\sqrt{2}} F_{12} |H|^2 \mp i\epsilon^{ij} \partial_i \left((\mathcal{D}_j H) H^\dagger \right) \, , \tag{1.130}$$

to obtain

$$T = \int_{\mathbb{C}} \Bigg[\frac{1}{2e^2} \left(F_{12} \mp \sqrt{\frac{\beta}{2}} e^2 \left(|H|^2 - \xi \right) \right)^2 + |(\mathcal{D}_1 \pm i\mathcal{D}_2) H|^2 \pm \frac{\sqrt{\beta} - 1}{\sqrt{2}} F_{12} |H|^2 \\ \mp \sqrt{\frac{\beta}{2}} \xi F_{12} \mp i\epsilon^{ij} \partial_i \left((\mathcal{D}_j H) H^\dagger \right) \Bigg] \, . \tag{1.131}$$

Now it is evident that something magical happens for $\beta = 1$, as the third term in the tension vanishes and the tension can become proportional to the integral over the magnetic flux, which is the topological charge of the vortex (the last term is just a boundary term, which is not important for finite-energy solutions in an infinite space-time configuration). $\beta = 1$ is the BPS limit which we already had found in the supersymmetric settings automatically, since it in fact corresponds to unbroken supersymmetry. In some later Chapters we will consider also non-BPS solutions which correspond in this simple model to $\beta \neq 1$. Choosing one of the signs corresponds to vortices and the other to anti-vortices (notice that the tension is never negative, but the flux can be so), and by convention we choose the upper sign to be vortices. The famous BPS equations are obtained simply by equating the first two squares of the above tension with zero

$$F_{12} - \frac{e^2}{\sqrt{2}} \left(|H|^2 - \xi \right) = 0 \, , \tag{1.132}$$

$$\bar{\mathcal{D}} H = 0 \, , \tag{1.133}$$

[4] Remember that we always normalize the generators to have $\mathrm{Tr}(t^2) = \frac{1}{2}$.

where $\bar{\mathcal{D}} \equiv (\mathcal{D}_1 + i\mathcal{D}_2)/2$. This gives the tension formula (by neglecting the boundary terms)

$$T_{\rm BPS} = -\frac{\xi}{\sqrt{2}} \int_{\mathbb{C}} F_{12} \; . \tag{1.134}$$

Normally, the vortices (BPS and non-BPS) are solved by choosing a radial Ansatz as

$$\begin{aligned} H &= \sqrt{\xi} h(r) e^{ik\theta} \; , \\ A_\theta &= -\frac{\sqrt{2}k}{r} a(r) \; , \end{aligned} \tag{1.135}$$

with k being the winding number. This leads to the BPS-equations

$$\frac{\partial_r a}{r} + \frac{\xi e^2}{2k}\left(h^2 - 1\right) = 0 \; , \tag{1.136}$$

$$\partial_r h - \frac{k}{r}(1-a)\, h = 0 \; , \tag{1.137}$$

To find the vortex solutions (for each k), the following boundary conditions should be imposed

$$r \to 0 \; : \quad a \propto r^2 \; , \quad h \propto r^k \; , \tag{1.138}$$

$$r \to \infty \; : \quad a \to 1 \; , \quad h \to 1 \; . \tag{1.139}$$

Inserting the Ansatz (1.135) into the tension formula (1.134) we obtain once again the already derived formula for the tension proportional simply to the topological charge

$$T_{\rm BPS} = 2\pi\xi k \; . \tag{1.140}$$

The two BPS equations (1.136), (1.137) can be combined to give

$$\nabla^2 \log h - \frac{\xi e^2}{2}\left(h^2 - 1\right) = 0 \; , \tag{1.141}$$

Notice the independence of k in the equation. This is in a radially symmetric Ansatz, so we easily see that Eq. (1.58) is exactly the same provided we take $N = 1$, $\Omega' = 1$, $\Omega_0 = r^{2k}$ and finally $\omega^{-1} = \frac{\xi}{r^{2k}} h^2$. The factor ξe^2 is the mass-squared parameter of the system. Setting the mass squared equal to two and substituting $h^2 \to e^{-\psi}$ we obtain the well-known Taubes equation

$$\nabla^2 \psi = 1 - e^{-\psi} \; . \tag{1.142}$$

This equation is not integrable. Taking a limit which puts the theory on the Coulomb branch, by sending $\sqrt{\xi} \to 0$ while keeping H constant, we obtain the Liouvilles equation

$$\nabla^2 \psi = -e^{-\psi} \; . \tag{1.143}$$

This equation has nothing to do with our vortex system, but it is integrable. The interesting point is that it has been found in the literature that the Abelian vortex, not on a plane (like here), but on a hyperbolic background geometry does indeed have the above equation of motion [169]. Recently, non-Abelian vortex moduli spaces have also been calculated analytically on this kind of background which provides powerful calculability [170].

Now if we take $\beta \neq 1$, we need to use the full equations of motion (1.127), (1.128) to which we apply the Ansatz (1.135) and obtain

$$\partial_r^2 h + \frac{1}{r}\partial_r h - \frac{k^2}{r^2}(1-a)^2 h - \frac{\beta\xi e^2}{2}\left(h^2 - 1\right) h = 0 \,, \tag{1.144}$$

$$\partial_r^2 a - \frac{1}{r}\partial_r a - \xi e^2 (a-1) h^2 = 0 \,. \tag{1.145}$$

From this system it is easy to read off the masses (around the Higgs vacuum $\langle h \rangle = \langle a \rangle = 1$) by making a perturbation which yields the following asymptotic functions

$$1 - h \sim e^{-m_h r} \,, \quad 1 - a \sim e^{-m_a r} \,, \tag{1.146}$$

where

$$m_h = \sqrt{\beta\xi} e \,, \quad m_a = \sqrt{\xi} e \,, \tag{1.147}$$

which also reveals the already mentioned meaning of the parameter β

$$\beta = \frac{m_h^2}{m_a^2} \,. \tag{1.148}$$

As the force associated with the scalar field is of an attractive nature while the force due to the magnetic field, that is the photon, makes up a pressure – a repulsive force, then we can understand very clearly what happens at critical coupling. Namely, something that happens very often in supersymmetric theories, the forces cancel at the critical coupling $\beta = 1$ and the vortices are BPS.

1.4.1 The wall vortex

In Refs. [171, 172] it was conjectured that the ANO vortex for large winding numbers which is equivalent to large magnetic flux, can be thought of as a domain wall between the Coulomb phase and the Higgs phase wrapped around the Coulomb phase. In Ref. [173] this conjecture was proven to hold by numerical methods. The renormalizable potential needed in four dimensional spacetime to break the $U(1)$ gauge symmetry is of the form shown in Fig. 1.1. The idea is simply balancing up the forces, the scalar field has a contractive force while the magnetic field and the energy density act as a pressure on the wrapped-up wall. We will now derive this vortex assuming that the scalar field vanishes inside the wall vortex and thus the magnetic field attains its maximal value in all the interior region. The physical intuition can be obtained from looking at the asymptotic expansions at the origin of the vortex and at infinity. Considering first the origin, we can see from Eq. (1.138) that in the limit of $k \to \infty$ the functions would presumably have the form

$$h \sim 0 \,, \quad a \sim r^2 \,, \tag{1.149}$$

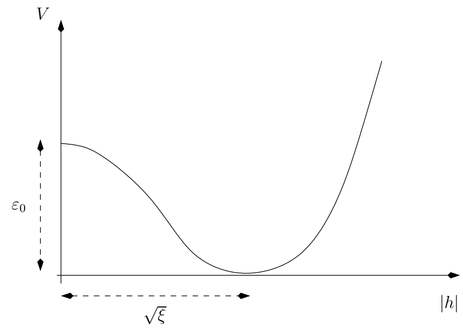


Figure 1.1: Potential for the Higgs field h which has an unstable Coulomb (symmetric) phase at $h = 0$ and a stable Higgs (asymmetric) phase at $|h| = \sqrt{\xi}$.

while if we look at Eq. (1.146) we have to consider the scales of the problem. The masses are fixed in the limit of $k \to \infty$, but the vortex radius grows: $R_V \propto k^\alpha$, where α is some power depending on the potential. This we will show shortly. Hence if we normalize the length scale by the vortex length, the exponentials vanish. Thus, the conjecture of the Refs. [171, 172] is that in the large magnetic flux limit $k \to \infty$, the profile functions become

$$h \sim \begin{cases} 0 & , \ r < R_V \ , \\ 1 & , \ r > R_V \ , \end{cases} \qquad a \sim \begin{cases} \left(\frac{r}{R_V}\right)^2 & , \ r < R_V \ , \\ 1 & , \ r > R_V \ . \end{cases} \tag{1.150}$$

First we consider the case that the potential is non-vanishing when the scalar field vanishes, which gives a tension contribution from the Coulomb phase. The maximal value of the magnetic field can be calculated to be $-\frac{\xi e^2}{\sqrt{2}}$. This contribution to the tension from the magnetic field for fixed flux k can be written as

$$\frac{1}{2e^2} \int_{\mathbb{C}} F_{12}^2 = \frac{\Phi^2}{e^2 \pi R^2} \ , \tag{1.151}$$

where the magnetic flux has been defined as

$$\Phi \equiv \frac{1}{\sqrt{2}} \int_{\mathbb{C}} F_{12} = -2\pi k \ , \quad k > 0 \ . \tag{1.152}$$

The tension of wrapping the wall around a circle is proportional to the circumference, while the contribution from the Coulomb phase is proportional to the area. Summing up all contributions, we can write the tension for fixed flux (k) as function of radius

$$T(R) = \frac{\Phi^2}{e^2 \pi R^2} + 2\pi R T_{\text{wall}} + \varepsilon_0 \pi R^2 \ , \tag{1.153}$$

and minimizing this expression with respect to the radius for large R, yields the wall-vortex radius

$$R_V = \frac{1}{\sqrt[4]{\varepsilon_0}} \sqrt{-\frac{\Phi}{\pi e}} \ , \tag{1.154}$$

which in turn yields the tension

$$T = -\xi\Phi = 2\pi\xi k \,, \tag{1.155}$$

where the large radius condition translates into the large magnetic flux limit as $R_V \propto \sqrt{k}$. It is denoted an MIT bag scaling. This is an interesting scaling, as it coincides with the scaling of the tension in the BPS case, namely $T \propto k$. Notice that so far nothing has been said about the coupling of the potential. This suggests that for all potentials with $\varepsilon_0 \neq 0$ the vortices will become BPS in the limit of sufficiently large magnetic flux. This is exactly what has been shown numerically in Ref. [173] for the Abelian-Higgs model.

Now in case that the vacuum energy ε_0 vanishes, a different scaling is present which yields the radius of the wall vortex

$$R_V = \sqrt[3]{\frac{\Phi^2}{e^2\pi T_{\rm wall}}} \,, \tag{1.156}$$

which in turn gives the tension

$$T = 3\sqrt[3]{\frac{\pi T_{\rm wall}^2 \Phi^2}{e^2}} \,. \tag{1.157}$$

Now the radius and the tension scale as $R_V \propto k^{2/3}$, $T \propto k^{2/3}$, respectively. This scaling is denoted the SLAC bag scaling.

In the case that the vacuum energy ε_0 is sufficiently small (much less than the maximum of the potential), an intermediate scaling will appear, namely the above mentioned SLAC bag scaling. However, at very large magnetic flux, the MIT bag scaling will take over and the tension will again become proportional to k. In the case with ε_0 strictly zero, the SLAC bag scaling is no longer intermediate but the true large magnetic flux scaling.

1.5 Chern-Simons

As already mentioned in the motivation, the Chern-Simons action is interesting for many reasons, among others that it is a topological term thus independent of the metric in the Lagrangian; it can naturally be embedded into non-relativistic theories by coupling it to the Schrödinger equation, which is a very useful property for condensed matter problems. Furthermore, it provides interesting features like fractional spin[5] and fractional charge [178, 179, 180, 181, 182, 183, 184, 185, 186]; and finally provides a topological mechanism of mass generation [187]. The price to pay is that it does only exist in odd space-time dimensions, thus $3, 5, \ldots$ and we will here consider only $3 = 2 + 1$.

The action can be written simply in terms of the integral of forms as

$$S_{\rm CS} = -\frac{\mu}{8\pi}\int \left(dA \wedge A - \frac{1}{3}A \wedge A \wedge A\right) \,. \tag{1.158}$$

[5]The fractional spin and statistics was first introduced in the NLσM with a Hopf term in the seminal paper [174] by Wilczek and later studied in subsequent works [175, 176, 177].

Note that there has not been used the space-time metric anywhere. Let us write the corresponding Yang-Mills action

$$S_{\mathrm{YM}} = -\frac{1}{4g_{\mathrm{YM}}^2}\int_{\mathbb{R}^3}\sqrt{\det g}\; g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\ . \tag{1.159}$$

If we take these two actions and calculate the corresponding contribution to the energy as the time-time component of the energy-momentum tensor, we get (in flat space-time)

$$E = \frac{1}{4g_{\mathrm{YM}}^2}\int_{\mathbb{R}^2}\left(F_{i0}^2 + F_{12}^2\right)\ . \tag{1.160}$$

for the Yang-Mills action, while the Chern-Simons action does not contribute at all.

The Lagrangian for Maxwell and Yang-Mills theories is gauge invariant by construction. The action for Chern-Simons theories does not a priori look gauge invariant (however that is not quite so). Let us first consider the Abelian Lagrangian

$$\mathcal{L} = -\frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\ , \tag{1.161}$$

and then make a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, the Lagrangian changes as

$$\delta\mathcal{L} = -\frac{\kappa}{4}\partial_\mu\left(\Lambda\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\right)\ , \tag{1.162}$$

which is nothing but a total derivative. Let us now consider a non-Abelian case, that is only a simple group, with the following Lagrangian

$$\mathcal{L} = -\frac{\mu}{8\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[A_\mu F_{\nu\rho} - \frac{i2}{3}A_\mu A_\nu A_\rho\right]\ , \tag{1.163}$$

and consecutively the gauge transformation $A_\mu \to UA_\mu U^{-1} + i(\partial_\mu U)U^{-1}$ which yields

$$\begin{aligned}\mathcal{L} \to \mathcal{L}\ &+ \frac{i\mu}{4\pi}\epsilon^{\mu\nu\rho}\partial_\mu\mathrm{Tr}\left[U^{-1}\left(\partial_\nu U\right)A_\rho\right]\\ &+ \frac{\mu}{12\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[U^{-1}\left(\partial_\mu U\right)U^{-1}\left(\partial_\nu U\right)U^{-1}\left(\partial_\rho U\right)\right]\ ,\end{aligned} \tag{1.164}$$

where the first term is a total derivative. The last term, however, is the winding number density of the group element U

$$\omega(U) \equiv \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[U^{-1}\left(\partial_\mu U\right)U^{-1}\left(\partial_\nu U\right)U^{-1}\left(\partial_\rho U\right)\right]\ , \tag{1.165}$$

which has the property $\Upsilon \equiv \int_{\mathbb{R}^3}\omega(U) \in \mathbb{Z}$. Thus we see that this last term gives rise to a change in the action

$$\delta S = 2\pi\mu\Upsilon\ . \tag{1.166}$$

Taking $\mu \in \mathbb{Z}$ as integers only, leaves the quantum amplitude gauge invariant. That is, the constant can be undone by a large gauge transformation.

Another very interesting fact about Chern-Simons theory is that it provides a topological mechanism for generating mass in a gauge theory [187]. Let us consider a neat Abelian example that combines Chern-Simons and Maxwell terms

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{4} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \,, \tag{1.167}$$

which has the equations of motion

$$\partial_\mu F^{\mu\sigma} - \frac{\kappa e^2}{4} \epsilon^{\mu\nu\sigma} F_{\mu\nu} = 0 \,. \tag{1.168}$$

Defining the dualized field strength as

$$\tilde{F}^\lambda \equiv \frac{1}{2} \epsilon^{\lambda\mu\nu} F_{\mu\nu} \,, \tag{1.169}$$

and inserting the equation of motion in terms of the dualized field strength into itself, leaves us with a Klein-Gordon-like equation for every component of this field

$$\left[\partial_\mu \partial^\mu + \left(\frac{\kappa e^2}{2}\right)^2\right] \tilde{F}^\lambda = 0 \,. \tag{1.170}$$

The same mass can also easily be calculated by obtaining the propagator of the gauge field and reading off the pole. This mass generation has nothing to do with spontaneous symmetry breaking (SSB). It can however be combined with the Higgs mechanism in the standard way, which will lead to two different massive modes in the gauge field.

1.5.1 Abelian Chern-Simons Higgs model

We will now make a short review of the Abelian Chern-Simons-Higgs model by coupling a complex scalar field to the Abelian Chern-Simons term

$$\mathcal{L} = -\frac{1}{4} \kappa \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + |\mathcal{D}_\mu H|^2 - \frac{1}{\kappa^2} \left(|H|^2 - \xi\right)^2 |H|^2 \,, \tag{1.171}$$

where the potential is tuned to its self-dual value [188, 189, 190, 191, 192]. There are two vacua in this Abelian theory, a symmetric vacuum where $\langle H \rangle = 0$ and an asymmetric vacuum with $|H| = \sqrt{\xi}$. The energy can be written as the time-time component of the energy-momentum tensor, which as we have already seen, does not include the Chern-Simons term

$$E = \int_{\mathbb{C}} \left[|\mathcal{D}_0 H|^2 + |\mathcal{D}_i H|^2 + \frac{1}{\kappa^2} \left(|H|^2 - \xi\right)^2 |H|^2\right] \,. \tag{1.172}$$

The equations of motion are

$$\mathcal{D}_\mu \mathcal{D}^\mu H + \frac{1}{\kappa^2} \left(|H|^2 - \xi\right) \left(3|H|^2 - \xi\right) H = 0 \,, \tag{1.173}$$

$$\kappa \epsilon^{\sigma\mu\nu} F_{\mu\nu} = -i 2\sqrt{2} \left[H^\dagger \mathcal{D}^\mu H - H \left(\mathcal{D}^\mu H\right)^\dagger\right] \,. \tag{1.174}$$

In particular, the last equation is the Gauss law (1.174). The electromagnetic field is non-dynamical since there is no Maxwell term and the time component of the gauge field is thus algebraically determined

$$A_0 = \frac{\kappa F_{12}}{2|H|^2} + \frac{i}{\sqrt{2}} \frac{H^\dagger \partial_0 H - H \partial_0 H^\dagger}{|H|^2} \ . \tag{1.175}$$

Hence, it will enter the Hamiltonian only through the covariant derivative.

Performing the Bogomol'nyi completion [193] we get

$$T = \int_{\mathbb{C}} \left[\left| \mathcal{D}_0 H - \frac{i}{\kappa} \left(|H|^2 - \xi \right) H \right|^2 + 4 \left| \bar{\mathcal{D}} H \right|^2 - \frac{\xi}{\sqrt{2}} F_{12} - i \epsilon^{ij} \partial_i \left(\left(\mathcal{D}_j H \right) H^\dagger \right) \right] \ , \tag{1.176}$$

where the complex covariant derivative is defined as $\bar{\mathcal{D}} \equiv (\mathcal{D}_1 + i\mathcal{D}_2)/2$. Thus, the bound on the energy is

$$E \geq -\xi \Phi \ , \tag{1.177}$$

where the magnetic flux has been defined as

$$\Phi \equiv \frac{1}{\sqrt{2}} \int_{\mathbb{C}} F_{12} = -2\pi k \ , \tag{1.178}$$

where $k \in \mathbb{Z}_+$ is the usual $U(1)$ winding number or the so-called vorticity. The BPS-equations can now readily be read off the expression for the tension yielding

$$\mathcal{D}_0 H - \frac{i}{\kappa} \left(|H|^2 - \xi \right) H = 0 \ , \tag{1.179}$$

$$\bar{\mathcal{D}} H = 0 \ , \tag{1.180}$$

which has to be accompanied by the Gauss law, being the $\sigma = 0$ component of Eq. (1.174) and can be written as

$$F_{12} = -\frac{i\sqrt{2}}{\kappa} \left[H^\dagger \mathcal{D}_0 H - H \left(\mathcal{D}_0 H \right)^\dagger \right] \ . \tag{1.181}$$

By combining the first BPS equation (1.179) with the Gauss law, we can finally obtain a system for the BPS Chern-Simons vortex which is similar to the ones for Yang-Mills or Maxwell (ANO) vortices, consisting of an equation for the magnetic field together with a "covariant holomorphy" condition

$$F_{12} = \frac{2\sqrt{2}}{\kappa^2} \left(|H|^2 - \xi \right) |H|^2 \ , \tag{1.182}$$

$$\bar{\mathcal{D}} H = 0 \ . \tag{1.183}$$

Now we can insert the radially symmetric Ansatz (1.135) and obtain

$$\frac{\partial_r a}{r} + \frac{2\xi^2}{k\kappa^2} \left(h^2 - 1 \right) h^2 = 0 \ , \tag{1.184}$$

$$\partial_r h - \frac{k}{r} \left(1 - a \right) h = 0 \ , \tag{1.185}$$

which in turn can be combined to a single second order differential equation for h

$$\nabla^2 \log h - \frac{2\xi^2}{\kappa^2}\left(h^2-1\right)h^2 = 0 \ . \tag{1.186}$$

The BPS mass can now easily be read off and it is

$$m_\kappa = \frac{2\xi}{\kappa} \ . \tag{1.187}$$

To find the vortex solutions (for each k), the following boundary conditions should be imposed

$$r \to 0 \ : \quad a \propto r^{2k+2} \ , \quad h \propto r^k \ , \tag{1.188}$$

$$r \to \infty \ : \quad a \to 1 \ , \quad h \to 1 \ . \tag{1.189}$$

1.5.2 Maxwell-Chern-Simons-Higgs model

Adding a Maxwell term to the Chern-Simons-Higgs Lagrangian, we have a richer system which will contain both Abrikosov-Nielsen-Olesen (ANO) vortices and Chern-Simons vortices in respective limits of the coupling constants [194, 195].

The self-dual Maxwell-Chern-Simons-Higgs Lagrangian reads [194, 195]

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\kappa\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + |D_\mu H|^2 + \frac{1}{2e^2}\left(\partial_\mu\phi\right)^2 \\ &-\frac{1}{2}\phi^2|H|^2 - \frac{e^2}{4}\left(|H|^2 - \frac{\kappa}{\sqrt{2}}\phi - \xi\right)^2 \ .\end{aligned} \tag{1.190}$$

Note that in order to obtain a self-dual theory, we have to introduce a neutral real scalar field ϕ. The interesting point to note about this theory is that it reduces to the Abelian-Higgs theory and the Chern-Simons theory in respective limits of the couplings e, κ. The theory has two degenerate vacua; a symmetric one with $\langle H\rangle = 0$ and $\langle\phi\rangle = -\sqrt{2}\xi/\kappa$ and an asymmetric one where $|\langle H\rangle| = \xi$ and $\langle\phi\rangle = 0$. Topological solitons exist in the asymmetric phase while so-called non-topological solitons exist in the symmetric phase, see e.g. Refs. [194, 195].

Let us explain the two limits giving the Abelian-Higgs model and the Chern-Simons-Higgs model, respectively. First we take $\kappa \to 0$ while keeping e fixed, which allows us to set $\phi = 0$. This yields exactly the Lagrangian (1.125). The other limit is obtained by sending $e \to \infty$ for fixed κ, which in turn allows us to integrate out the neutral scalar field ϕ as

$$\phi = \frac{\sqrt{2}}{\kappa}\left(|H|^2 - \xi\right) \ . \tag{1.191}$$

This gives us the Lagrangian (1.171).

Let us now write the energy of the vortex system using the Bogomol'nyi completion

$$
\begin{aligned}
E = \int_{\mathbb{C}} \Bigg[& \frac{1}{2e^2}\left(F_{0i}\right)^2 + \frac{1}{2e^2}\left(F_{12}\right)^2 + \frac{1}{2e^2}\left(\partial_0\phi\right)^2 + \frac{1}{2e^2}\left(\partial_i\phi\right)^2 \\
& + |\mathcal{D}_0 H|^2 + |\mathcal{D}_i H|^2 + \frac{1}{2}\phi^2|H|^2 + \frac{e^2}{4}\left(|H|^2 - \frac{\kappa}{\sqrt{2}}\phi - \xi\right)^2 \Bigg] \\
= \int_{\mathbb{C}} \Bigg[& \frac{1}{2e^2}\left(F_{12} - \frac{e^2}{\sqrt{2}}\left(|H|^2 - \frac{\kappa}{\sqrt{2}}\phi - \xi\right)\right)^2 + \frac{1}{2e^2}\left(F_{0i} + \partial_i\phi\right)^2 \\
& + 4\left|\bar{\mathcal{D}}H\right|^2 + \left|\mathcal{D}_0 H - \frac{i}{\sqrt{2}}\phi H\right|^2 + \frac{1}{2e^2}\left(\partial_0\phi\right)^2 \\
& - \frac{\xi}{\sqrt{2}}F_{12} - \frac{1}{e^2}\partial_i\left(F_{0i}\phi\right) - i\epsilon^{ij}\partial_i\left(\left(\mathcal{D}_j H\right)H^\dagger\right) \Bigg] \,. \qquad (1.192)
\end{aligned}
$$

It is not hard to understand that the above demonstrated limits manifest themselves similarly in the BPS equations (see e.g. Ref. [7]), which are

$$F_{12} - \frac{e^2}{\sqrt{2}}\left(|H|^2 - \frac{\kappa}{\sqrt{2}}\phi - \xi\right) = 0 \,, \qquad (1.193)$$

$$F_{0i} + \partial_i\phi = 0 \,, \qquad (1.194)$$

$$\bar{\mathcal{D}}H = 0 \,, \qquad (1.195)$$

$$\mathcal{D}_0 H - \frac{i}{\sqrt{2}}\phi H = 0 \,, \qquad (1.196)$$

$$\partial_0\phi = 0 \,, \qquad (1.197)$$

and they should be accompanied by the Gauss law

$$\frac{1}{e^2}\partial_\mu F^{\mu\sigma} - \frac{\kappa}{4}\epsilon^{\mu\nu\sigma}F_{\mu\nu} - \frac{i}{\sqrt{2}}\left(H^\dagger\mathcal{D}^\sigma H - \left(\mathcal{D}^\sigma H\right)^\dagger H\right) = 0 \,. \qquad (1.198)$$

It is easy to see that static solutions are found by setting $A_0 = \phi$. The vortex system is then found with this condition together with the combination of Eq. (1.196) and the Gauss law, together of course with the usual Eq. (1.195). This leaves us with

$$\bar{\mathcal{D}}H = 0 \,, \qquad (1.199)$$

$$F_{12} - \frac{e^2}{\sqrt{2}}\left(|H|^2 - \frac{\kappa}{\sqrt{2}}\phi - \xi\right) = 0 \,, \qquad (1.200)$$

$$\frac{1}{e^2}\partial_i^2\phi + \frac{\kappa}{2}F_{12} - \phi|H|^2 = 0 \,. \qquad (1.201)$$

For completeness, let us demonstrate the two limits in this vortex system. Taking $\kappa \to 0$, we see that the first two equations (1.199)-(1.200) describe the ANO vortex (1.132) while the third equation has just a trivial solution (and all other solutions are gauge equivalent to the trivial one). Taking the other limit, by sending $e \to \infty$ we find from the second equation the value of ϕ which is being integrated out identically as in Eq. (1.191) while the last equation (1.201) becomes the one describing the Chern-Simons vortex (1.182).

1.5.3 Non-Abelian Chern-Simons model

By the same token as in the last Section we can extend the Maxwell-Chern-Simons model to a non-Abelian generalization by considering a gauge group $G = U(1) \times G'$ instead of just $G = U(1)$. This will be the Yang-Mills-Chern-Simons-Higgs theory. We are considering the following $\mathcal{N} = 2$ supersymmetric theory (viz. with 4 supercharges) in $d = 2 + 1$ dimensions with the gauge group $G = U(1) \times G'$, where G' is a simple group. The bosonic part of the Lagrangian density reads

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YMCSH}} = & -\frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{4e^2}\left(F^0_{\mu\nu}\right)^2 - \frac{\mu}{8\pi}\epsilon^{\mu\nu\rho}\left(A^a_\mu \partial_\nu A^a_\rho - \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\right) \\
& - \frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho} A^0_\mu \partial_\nu A^0_\rho \qquad (1.202) \\
& + \frac{1}{2g^2}\left(\mathcal{D}_\mu \phi^a\right)^2 + \frac{1}{2e^2}\left(\partial_\mu \phi^0\right)^2 + \mathrm{Tr}\left(\mathcal{D}_\mu H\right)\left(\mathcal{D}^\mu H\right)^\dagger - \mathrm{Tr}\left|\phi H - H m\right|^2 \\
& - \frac{g^2}{2}\left(\mathrm{Tr}\left(H H^\dagger t^a\right) - \frac{\mu}{4\pi}\phi^a\right)^2 - \frac{e^2}{2}\left(\mathrm{Tr}\left(H H^\dagger t^0\right) - \frac{\kappa}{4\pi}\phi^0 - \frac{1}{\sqrt{2N}}\xi\right)^2 ,
\end{aligned}
$$

where $a = 1, \ldots, \dim(G')$ for the non-Abelian group, the index 0 is for the Abelian group and $\alpha = 0, 1, \ldots, \dim(G')$. We are now considering non-Abelian Chern-Simons theories with an Abelian factor in the gauge symmetry. Therefore we have rescaled the Chern-Simons coupling constant $\kappa \longrightarrow \kappa/4\pi$. Hence equal coupling (level) for the non-Abelian and Abelian Chern-Simons interactions is now corresponding to $\kappa = \mu$. We use the conventions

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right] , \qquad (1.203)
$$

$$
\mathcal{D}_\mu H = \left(\partial_\mu + i A_\mu\right) H , \qquad (1.204)
$$

$$
\mathcal{D}_\mu \phi = \partial_\mu \phi + i\left[A_\mu, \phi\right] . \qquad (1.205)
$$

$A_\mu = A^\alpha_\mu t^\alpha$ is the gauge potential, $F_{\mu\nu}$ is the field strength, ϕ is an adjoint scalar field which we can take to be real and finally H is a color-flavor matrix of dimension $N \times N_{\mathrm{F}}$ of N_{F} matter fields. We will define $N \equiv \dim(R_{G'})$ but for simplicity we choose the representation $R_{G'}$ as the fundamental one of G'. We are using the following normalization of the generators

$$
t^0 = \frac{\mathbf{1}_N}{\sqrt{2N}} , \quad \mathrm{Tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab} . \qquad (1.206)
$$

There are four coupling constants entering our game at this point; $e \in \mathbb{R}$ is the Abelian coupling of the Yang-Mills kinetic term (Maxwell), $g \in \mathbb{R}$ the is the coupling for the semi-simple part of the Yang-Mills kinetic term, which corresponds to G'. $\kappa \in \mathbb{R}$ is the Abelian coupling of the Chern-Simons term while $\mu \in \mathbb{Z}$ are solely integers to render the non-Abelian Chern-Simons action gauge invariant up to large gauge transformations [196]. ξ is a Fayet-Iliopoulos parameter. Finally, m is a mass matrix which we will set to zero in the following.

The tension of the system reads

$$
\begin{aligned}
T = \int_{\mathbb{C}} \Bigg[& \frac{1}{2g^2}\left(F^a_{0i}\right)^2 + \frac{1}{2g^2}\left(F^a_{12}\right)^2 + \frac{1}{2e^2}\left(F^0_{0i}\right)^2 + \frac{1}{2e^2}\left(F^0_{12}\right)^2 \\
& + \frac{1}{2g^2}\left(\mathcal{D}_0\phi^a\right)^2 + \frac{1}{2g^2}\left(\mathcal{D}_i\phi^a\right)^2 + \frac{1}{2e^2}\left(\partial_0\phi^0\right)^2 + \frac{1}{2e^2}\left(\partial_i\phi^0\right)^2 \\
& + \mathrm{Tr}\left(\mathcal{D}_0 H\right)^\dagger\left(\mathcal{D}_0 H\right) + \mathrm{Tr}\left(\mathcal{D}_i H\right)^\dagger\left(\mathcal{D}_i H\right) + \mathrm{Tr}\left|\phi H - Hm\right|^2 \\
& + \frac{g^2}{2}\left(\mathrm{Tr}\left(HH^\dagger t^a\right) - \frac{\mu}{4\pi}\phi^a\right)^2 + \frac{e^2}{2}\left(\mathrm{Tr}\left(HH^\dagger t^0\right) - \frac{\kappa}{4\pi}\phi^0 - \frac{1}{\sqrt{2N}}\xi\right)^2 \Bigg] .
\end{aligned} \tag{1.207}
$$

Using the following useful identity

$$
\mathcal{D}_i H\left(\mathcal{D}_i H\right)^\dagger = \left(D_1 \pm i\mathcal{D}_2\right) H\left[\left(\mathcal{D}_1 \pm i\mathcal{D}_2\right) H\right]^\dagger \mp F^\alpha_{12} t^\alpha HH^\dagger \mp i\epsilon^{ij}\partial_i\left(\left(\mathcal{D}_j H\right) H^\dagger\right) , \tag{1.208}
$$

we perform a standard Bogomol'nyi completion

$$
\begin{aligned}
T = \int_{\mathbb{C}} \Bigg[& \frac{1}{2g^2}\left(F^a_{12} - g^2\left(\mathrm{Tr}\left(HH^\dagger t^a\right) - \frac{\mu}{4\pi}\phi^a\right)\right)^2 + \frac{1}{2g^2}\left(F^a_{0i} + \mathcal{D}_i\phi^a\right)^2 \\
& + \frac{1}{2e^2}\left(F^0_{12} - e^2\left(\mathrm{Tr}\left(HH^\dagger t^0\right) - \frac{\kappa}{4\pi}\phi^0 - \frac{\xi}{\sqrt{2N}}\right)\right)^2 + \frac{1}{2e^2}\left(F^0_{0i} + \partial_i\phi^0\right)^2 \\
& + 4\mathrm{Tr}\left|\bar{\mathcal{D}} H\right|^2 + \mathrm{Tr}\left|\mathcal{D}_0 H - i\left(\phi H - Hm\right)\right|^2 + \frac{1}{2g^2}\left(\mathcal{D}_0\phi^a\right)^2 + \frac{1}{2e^2}\left(\partial_0\phi^0\right)^2 \\
& - \frac{\xi}{\sqrt{2N}} F^0_{12} + i\mathrm{Tr}\left[\left(H^\dagger\mathcal{D}_0 H - \left(\mathcal{D}_0 H\right)^\dagger H\right) m\right] \\
& - \frac{1}{g^2}\partial_i\left(F^a_{0i}\phi^a\right) - \frac{1}{e^2}\partial_i\left(F^0_{0i}\phi^0\right) - i\epsilon^{ij}\mathrm{Tr}\,\partial_i\left(\left(\mathcal{D}_j H\right) H^\dagger\right) \Bigg] ,
\end{aligned} \tag{1.209}
$$

where we have used Gauss' law. Focusing on BPS-solutions, the tension is given by the saturated Bogomol'nyi bound

$$
\begin{aligned}
T_{\mathrm{BPS}} &= -\frac{\xi}{\sqrt{2N}}\int_{\mathbb{C}} F^0_{12} + i\mathrm{Tr}\int_{\mathbb{C}}\left[\left(H^\dagger\mathcal{D}_0 H - \left(\mathcal{D}_0 H\right)^\dagger H\right) m\right] , \\
&= 2\pi\xi\nu + \mathrm{Tr}\left(Qm\right) ,
\end{aligned} \tag{1.210}
$$

with ν being the $U(1)$ winding and

$$
Q \equiv i\int_{\mathbb{C}}\left[\left(H^\dagger\mathcal{D}_0 H - \left(\mathcal{D}_0 H\right)^\dagger H\right)\right] , \tag{1.211}
$$

are the Noether charges associated with the breaking of color-flavor symmetry. The BPS-equations are readily read to be

$$\bar{\mathcal{D}}H = 0 \,, \tag{1.212}$$

$$F_{12}^a - g^2\left(\mathrm{Tr}\left(HH^\dagger t^a\right) - \frac{\mu}{4\pi}\phi^a\right) = 0 \,, \tag{1.213}$$

$$F_{12}^0 - e^2\left(\mathrm{Tr}\left(HH^\dagger t^0\right) - \frac{\kappa}{4\pi}\phi^0 - \frac{\xi}{\sqrt{2N}}\right) = 0 \,, \tag{1.214}$$

$$F_{0i}^a + \mathcal{D}_i\phi^a = 0 \,, \tag{1.215}$$

$$F_{0i}^0 + \partial_i\phi^0 = 0 \,, \tag{1.216}$$

$$\mathcal{D}_0 H - i\left(\phi H - Hm\right) = 0 \,, \tag{1.217}$$

$$\mathcal{D}_0\phi^a = 0 \,, \tag{1.218}$$

$$\partial_0\phi^0 = 0 \,, \tag{1.219}$$

which have to be accompanied by the Gauss law

$$\frac{1}{g^2}\mathcal{D}_i F_{0i}^e - \frac{\mu}{4\pi}F_{12}^e - \frac{1}{g^2}f^{eab}\phi^a\mathcal{D}_0\phi^b - i\mathrm{Tr}\left(H^\dagger t^e\mathcal{D}_0 H - \left(\mathcal{D}_0 H\right)^\dagger t^e H\right) = 0 \,, \tag{1.220}$$

$$\frac{1}{e^2}\partial_i F_{0i}^0 - \frac{\kappa}{4\pi}F_{12}^0 - i\mathrm{Tr}\left(H^\dagger t^0\mathcal{D}_0 H - \left(\mathcal{D}_0 H\right)^\dagger t^0 H\right) = 0 \,, \tag{1.221}$$

Considering now $m = 0$, we can consistently choose $\phi = A_0$ in temporal gauge (appropriate for static solutions). We are left with the following system

$$\bar{\mathcal{D}}H = 0 \,, \tag{1.222}$$

$$F_{12}^a - g^2\left(\mathrm{Tr}\left(HH^\dagger t^a\right) - \frac{\mu}{4\pi}\phi^a\right) = 0 \,, \tag{1.223}$$

$$F_{12}^0 - e^2\left(\mathrm{Tr}\left(HH^\dagger t^0\right) - \frac{\kappa}{4\pi}\phi^0 - \frac{\xi}{\sqrt{2N}}\right) = 0 \,, \tag{1.224}$$

$$\frac{1}{g^2}\mathcal{D}_i^2\phi^a + \frac{\mu}{4\pi}F_{12}^a - \mathrm{Tr}\left(\left\{\phi, t^a\right\}HH^\dagger\right) = 0 \,, \tag{1.225}$$

$$\frac{1}{e^2}\partial_i^2\phi^0 + \frac{\kappa}{4\pi}F_{12}^0 - \mathrm{Tr}\left(\left\{\phi, t^0\right\}HH^\dagger\right) = 0 \,, \tag{1.226}$$

where $\phi \equiv \phi^\alpha t^\alpha$. Choosing for instance $G' = SU(N)$, we have the following system

$$\bar{\mathcal{D}}H = 0 \,, \tag{1.227}$$

$$F_{12}^a t^a - \frac{g^2}{2}\left(HH^\dagger - \frac{1}{N}\mathrm{Tr}\left(HH^\dagger\right)\mathbf{1}_N - \frac{\mu}{2\pi}\phi^a t^a\right) = 0 \,, \tag{1.228}$$

$$F_{12}^0 t^0 - \frac{e^2}{2}\left(\frac{1}{N}\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right)\mathbf{1}_N - \frac{\kappa}{2\pi}\phi^0 t^0\right) = 0 \,, \tag{1.229}$$

$$\frac{1}{g^2}\mathcal{D}_i^2\phi^a t^a + \frac{\mu}{4\pi}F_{12}^a t^a - \frac{1}{2}\left\{HH^\dagger, \phi\right\} + \frac{1}{N}\mathrm{Tr}\left(HH^\dagger\phi\right)\mathbf{1}_N = 0 \,, \tag{1.230}$$

$$\frac{1}{e^2}\partial_i^2\phi^0 t^0 + \frac{\kappa}{4\pi}F_{12}^0 t^0 - \frac{1}{N}\mathrm{Tr}\left(HH^\dagger\phi\right)\mathbf{1}_N = 0 \,. \tag{1.231}$$

Let us close this Section with a discussion of the various models that can be reached as limits of this wonderful theory just presented.

- Maxwell-Yang-Mills-Higgs ($\kappa \to 0$ and $\mu \to 0$, $m = 0$): There exist a Higgs vacuum: $H = \mathbf{1}_N \sqrt{\xi/N}$ and $\phi = 0$ (it could be of a more complicated form depending on the gauge symmetry). The vortices in this model are the non-Abelian generalization of the ANO vortex.

  The mass spectrum is of the form

$$m_e^2 = \frac{\xi e^2}{N} \ , \quad m_g^2 = \frac{\xi g^2}{N} \ , \tag{1.232}$$

  which due to supersymmetry are the masses for both the vector bosons and the Higgs bosons, where m_e is for the trace-part and m_g is for the traceless part.

- Maxwell-Yang-Mills-Chern-Simons ($H \equiv 0$, $m = 0$)

  If we eliminate completely the Higgs fields, the model reduces to the Maxwell-Yang-Mills-Chern-Simons model (which in turn allows us to further eliminate the FI parameter by a change of variables). The vector multiplet here acquires a mass by the topological argument presented previously in the Abelian case

$$m_\kappa = \frac{\kappa e^2}{4\pi} \ , \quad m_\mu = \frac{\mu g^2}{4\pi} \ . \tag{1.233}$$

  where m_κ is for the trace-part while m_μ is for the traceless part of the gauge fields.

- Chern-Simons-Higgs ($e \to \infty$ and $g \to \infty$)

  The Maxwell and Yang-Mills kinetic terms disappear in this limit and the theory reduces to the Chern-Simons-Higgs model. Furthermore, the adjoint scalar fields ϕ^α become non-dynamical fields and can be integrated out as follows

$$\phi^0 = \frac{4\pi}{\kappa}\left(\mathrm{Tr}\left(HH^\dagger t^0\right) - \frac{\xi}{\sqrt{2N}}\right) \ , \quad \phi^a = \frac{4\pi}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right) \ . \tag{1.234}$$

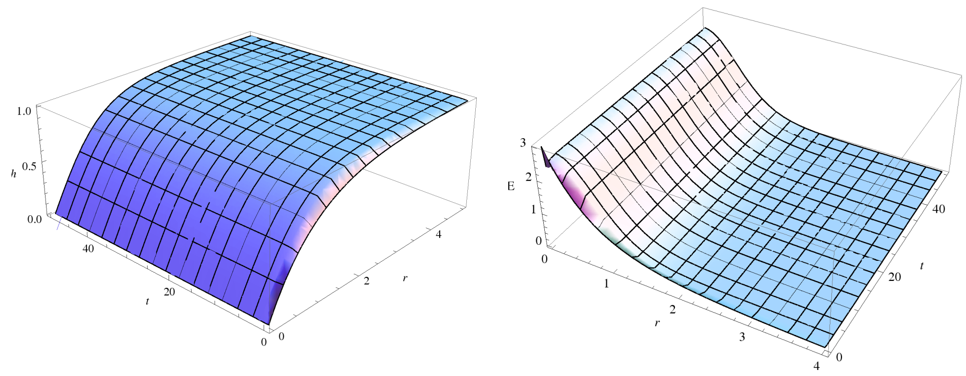
  The model then acquires a sixth order potential. There exist two vacua in this model. One is in a symmetric phase where $\langle H \rangle = 0$ and the symmetry remains unbroken in the vacuum. The vector multiplets are decoupled, so the Higgs fields are the only dynamical degrees of freedom. Their masses are

$$m_H = \frac{2\pi\xi}{\kappa}. \tag{1.235}$$

  The other vacuum is in the Higgs phase where the gauge symmetry is completely broken. The longitudinal part of the gauge fields acquire a mass via the Higgs mechanism. By supersymmetry the mass of the Higgs fields is the same as that of gauge fields

$$m_{\kappa\infty} = \frac{2\pi\xi}{\kappa} \ , \quad m_{\mu\infty} = \frac{2\pi\xi}{\mu} \ . \tag{1.236}$$

  In the symmetric phase of the Abelian case, it is known that non-topological vortices do in fact exist.



(a) (b)

Figure 1.2: (a) The profile function h and (b) the energy density for the BPS ANO vortex found by the cooling method, both shown as function of the radial coordinate r and the fictitious time t. Notice the quick convergence.

1.6 Numerical methods

1.6.1 Cooling method

The method that will prove most powerful for solving the equations that we have encountered in the studies of vortices etc., is the cooling method which is also called the relaxation method. It is convenient because the method by construction seeks towards an extremum of the action, which is exactly what we want; the solution to the equations of motion. Let us just mention the technique briefly by solving an easy example, the BPS ANO vortex. Take Eq. (1.141) and set for convenience $\xi e^2 = 2$. Then we add a fictitious time-dependence and equate the $\nabla^2 + X$ operator with the ∂_t operator. In this simple example we have

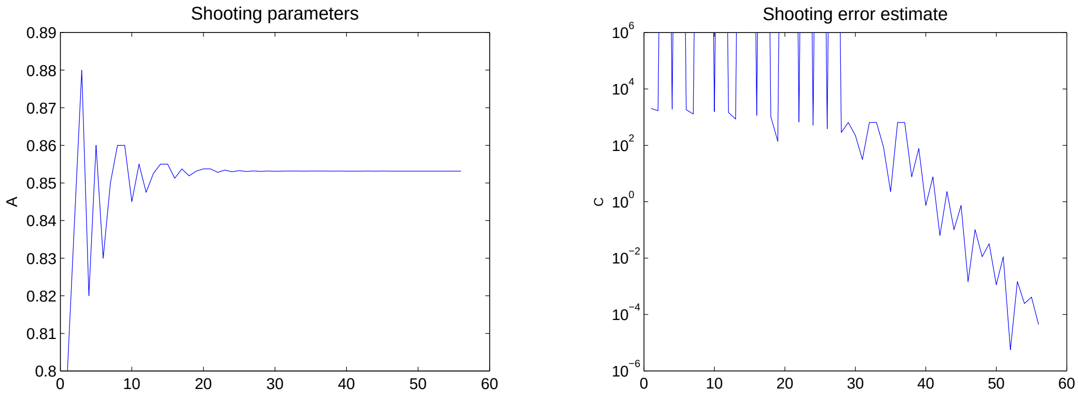
$$\partial_r^2 h + \frac{1}{r}\partial_r h - \frac{(\partial_r h)^2}{h} - \left(h^2 - 1\right) h = \partial_t h \ , \quad h = h(t,r) \ . \tag{1.237}$$

The method is very fast and in a short time t, the solution will be found. In Fig. 1.2 we show the field h and the energy distribution in a three-dimensional graph as function of the radial coordinate r and the fictitious time t. It is easily seen that the surface does not change as function of time after $t > 10$, which means that a solution has been obtained.

1.6.2 Shooting method

Let us briefly describe another method that we have used in some cases, which is the shooting method. Let us take the same example as in the last Subsection, namely the BPS ANO vortex. We take again $\xi e^2 = 2$ and rewrite the equations as a system into the form

$$\partial_r \begin{pmatrix} a \\ h \end{pmatrix} = \begin{pmatrix} -r\left(h^2 - 1\right) \\ \frac{1}{r}\left(1 - a\right) h \end{pmatrix} \ . \tag{1.238}$$



(a) (b)

Figure 1.3: (a) The shooting parameters and (b) the error function for the BPS ANO vortex used by a steepest descent method to find the solution.

The boundary conditions at $r \to \infty$ are $a \to 1$ and $h \to 1$ with exponential corrections, which means that we can search for the solution by minimizing the following function

$$C(a_\infty, h_\infty) = |a_\infty - 1|^2 + |h_\infty - 1|^2 \ , \tag{1.239}$$

where we have defined $a_\infty = \lim_{r \to r_{\max}} a(r)$ with $r_{\max}$ being the finite cut-off in the numerical simulation (and the definition is similar for $h(r)$). Using the equations of motion and inserting a power series, we find that for the BPS case, which we are studying, the first terms in the limit $r \to 0$ in the expansion are

$$a \simeq \frac{1}{2} r^2 \ , \tag{1.240}$$

$$h \simeq A r \ , \tag{1.241}$$

where A is our shooting parameter. For the non-BPS ANO vortex, we need two shooting parameters and for non-Abelian cases, we also have a more complex parameter space. The method nevertheless is exactly the same. In Fig. 1.3a we show the shooting parameters used by the steepest descent method to minimize the function C which finds the value $A = 0.853152$, while in Fig. 1.3b we show the corresponding error function C. The profile functions a, h are shown in Fig. 1.4.

1.7 Vortex dynamics and effective world-sheet theory

Let us briefly discuss the quantum effects of the vortices in the $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory with $N_{\rm F} = N$ hypermultiplets. There is, as already mentioned, both a $U(N)$ vector-multiplet and an adjoint chiral multiplet in the theory. We consider having a non-negative FI parameter $\xi > 0$, i.e. we put the theory on the Higgs branch, but we do not break $\mathcal{N} = 2$ here. This is also an interesting topic, especially related to the questions

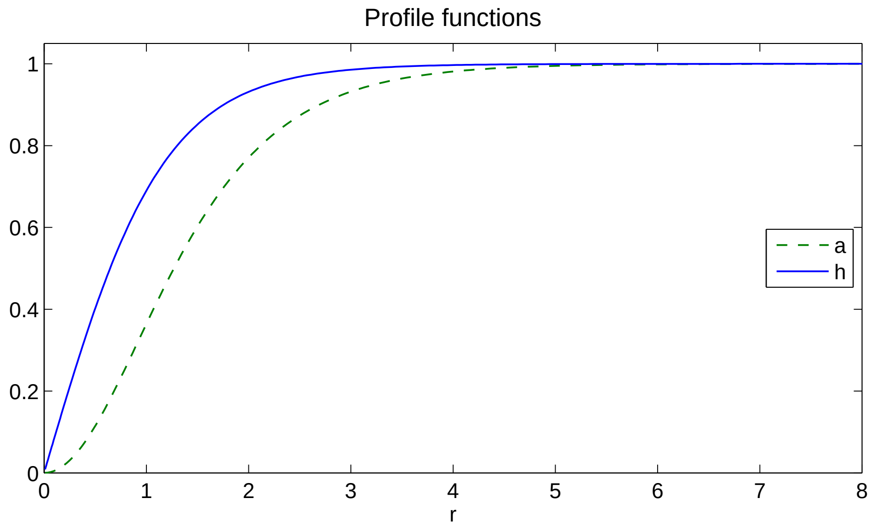


Figure 1.4: (a) The profile functions h and a for the BPS ANO vortex found using the shooting method. We have set $\xi e^2 = 2$ and $r_{\rm max} = 10$.

concerning non-Abelian monopoles. As already mentioned in the first Chapter, an exact correspondence between the BPS spectra of this theory and an $\mathcal{N} = (2,2)$ supersymmetric $\mathbb{C}P^{N-1}$ σ model has been noticed in Ref. [107] and explained via the non-Abelian vortex solutions in the Refs. [8, 106, 104]. A very important ingredient of the spectrum are the monopoles. Considering a high-energy theory with a simple gauge group (without an overall $U(1)$ factor), it can be broken down as $SU(N+1) \to U(1) \times SU(N)$ giving rise to our model as the low-energy theory. There will be various monopoles associated with this type of breaking, for instance, very heavy monopoles which are associated with a weight vector of the (dual) group in $SU(N)$ on which vortices can end [105]. We will not discuss this kind of breaking further in this Section, but concentrate on the lighter monopoles which do appear in the BPS spectrum of the theory as confined monopoles and as we will see, as kinks on the low-energy effective world-sheet theory.

Let us first consider how the low-energy effective world-sheet action comes about. Choosing an Ansatz by embedding the ANO vortex solution in the upper-left corner of the matrices of $U(N)$ as

$$U^{-1} H U = \mathrm{diag}\left(h(r), f(r)\mathbf{1}_{N-1}\right) \ , \tag{1.242}$$

$$U^{-1} A_\theta^a t^a U = \frac{b(r)-1}{r} T \ , \quad T \equiv \frac{1}{\sqrt{2N(N-1)}} \mathrm{diag}\big(-N+1, \mathbf{1}_{N-1}\big) \ , \tag{1.243}$$

$$U^{-1} A_\theta^0 t^0 U = \frac{1-a(r)}{r} t^0 \ , \tag{1.244}$$

we can define the orientational vector from $U \in G_{C+F}$ i.e. a (global) color-flavor transformation

$$U T U^{-1} = -n n^\dagger + \frac{\mathbf{1}_N}{\sqrt{2N(N-1)}} \ , \tag{1.245}$$

with n being a row vector in the fundamental representation of $SU(N)$, which has to obey the following constraint

$$n^\dagger n = 1 \,, \tag{1.246}$$

while the combination $nn\dagger$ is an Hermitian matrix. This vector will furthermore turn out to be the coordinate of the σ model. Its components will be assumed to be slowly varying functions of x^3 (i.e. along the direction of the vortex world-sheet) and time x^0. The σ model describes the fluctuations of the orientational modes along the vortex, i.e. zero-modes and thus there is no potential term in the effective action. The non-linear formulation of the σ model reads

$$S_{\text{world-sheet}} = 2\beta \int_{\mathbb{R}^2} \left[\partial_\alpha n^\dagger \partial^\alpha n + \left(n^\dagger \partial_\alpha n\right)^2\right] \,, \tag{1.247}$$

which is equivalent to a linear gauged $U(1)$ formulation which has the former action as its strong coupling limit ($e_{2d} \to \infty$)

$$\begin{aligned} S_{\text{linear gauged}} = \int_{\mathbb{R}^2} \Bigg[& -\frac{1}{4e_{2d}^2} F_{\alpha\beta} + 2\beta \mathcal{D}_\alpha n^\dagger \mathcal{D}^\alpha n + \frac{1}{e_{2d}^2} |\partial_\alpha \sigma|^2 \\ & - 4\beta |\sigma|^2 n^\dagger n - 2e_{2d}^2 \beta^2 \left(n^\dagger n - 1\right)^2 \Bigg] \,, \end{aligned} \tag{1.248}$$

where it is understood that the integrals are over the x^3 coordinate and time. σ is a complex scalar and β is interestingly found to be related to the coupling of the four-dimensional theory as

$$\beta = \frac{2\pi}{g} \,, \tag{1.249}$$

in the BPS case, where we have set $e = g$ in the four-dimensional gauge theory. The physical UV cut-off for the σ model should be taken as $\sqrt{\xi} g$ while below this value, the coupling β runs due to the two-dimensional renormalization group (RG) flow

$$4\pi\beta = N \ln\left(\frac{\mu}{\Lambda_\sigma}\right) \,, \tag{1.250}$$

where μ is the energy-scale while Λ_σ is the dynamical scale of the σ model, which turns out to be identified with the strong-coupling scale of the mother gauge theory $\Lambda_\sigma = \Lambda_{SU(N)}$. Below this scale, the four-dimensional gauge coupling is frozen and β takes over.

In this current case we are reviewing now, we are in a good shape due to the fact that Witten has solved the $\mathbb{C}P^{N-1}$ model in the large N limit [197]. The calculation can be done due to the non-perturbatively generated exact superpotential by a summation of all-orders instanton effects. Classically, the orientational field n can take an arbitrary direction and one might expect a SSB taking place which in turn would give rise to massless Goldstone modes. This is not the case. In fact it is not allowed by the theorem due to Coleman [198] in two dimensions. What happens is that due to quantum effects, the symmetry is restored and the condition on the norm of the field n gets in some sense relaxed. Witten showed in his seminal paper [197] that this model has indeed N vacua, all with a mass gap. The moduli space of continuous solutions is gone and the fields n become massive. At strong coupling, the chiral condensate is the order parameter of the model and its value differs in different vacua. Furthermore, there are two anomalies taking place here, first the chiral anomaly breaks the chiral $U(1)$ symmetry down to $\mathbb{Z}_{2N}$ while the fermion condensate – the order parameter, breaks it further down to $\mathbb{Z}_2$. An interesting consequence of the fact that quantumly, the field n does not take a particular direction, is that the orientation becomes completely smeared out. That is, the expectation value vanishes $\langle n \rangle = 0$. Hence, we are not allowed to identify this quantum vortex in a single vacuum, with the classical one which uses a definite Cartan generator in the group.

Very interestingly, there has been made a mirror model of the $\mathbb{C}P^{N-1}$ σ model, in which the description is in the form of a Coulomb gas of instantons [199] which in turn is equivalent to an $\mathcal{N} = 2$ affine Toda theory [200, 201, 202].

Let us return to the already mentioned monopoles in the four-dimensional theory. First we will set the FI parameter to zero $\xi = 0$ and thus enter the Coulomb branch, however, we now want to turn on different masses, breaking the gauge symmetry with the following superpotential

$$\mathcal{W} = \sqrt{2} \sum_{i}^{N_{\rm F}} \tilde{Q}_i \left(\Phi + m_i\right) Q_i \ . \tag{1.251}$$

Let us furthermore define the following antisymmetric matrix $\Delta m_{ij} \equiv m_i - m_j$. Until now we have always assumed this matrix to be zero, breaking the gauge symmetry completely, however, if all the parameters m_i are different, we have the following breaking

$$U(N) \xrightarrow{\Delta m} U(1)^N \ , \tag{1.252}$$

supporting 't Hooft-Polyakov monopoles in the Coulomb phase – in fact $N - 1$ different ones. Now consider turning back on the FI parameter $\xi \neq 0$. Vortex strings can be made, however, the non-Abelian nature is absent – the moduli space has been lifted and there are no continuous orientational moduli. That is, the vector n has N discrete possibilities which resembles a completely Abelian nature. The vortices must be present and furthermore they have to confine the flux due to the monopoles. An amazing result found by Shifman-Yung [8] is that the confined monopoles in the bulk theory manifest themselves as kinks in the $\mathbb{C}P^{N-1}$ model. We will not provide the demonstration here, but simply show an intuitive picture of what happens, see Fig. 1.5. In fact, it has been shown that the masses of the monopoles coincide with those of the kinks on the σ model. The mass turns out to be $2\beta\Delta m_{ij}$ and this is also welcomed by the fact that masses of the BPS monopole solutions cannot depend on the non-holomorphic quantity ξ. The last picture of Fig. 1.5 is the situation in which the mass differences are being turned off and we are entering a highly quantum regime. Here the mass of the kinks and thus the monopoles becomes $2\Lambda_\sigma/\pi$.

We have discussed the monopoles of the BPS spectrum of the two theories, the four-dimensional one and the two-dimensional σ model. The rest of the spectrum does in fact also coincide in these two theories as already mentioned. For instance, in the perturbative

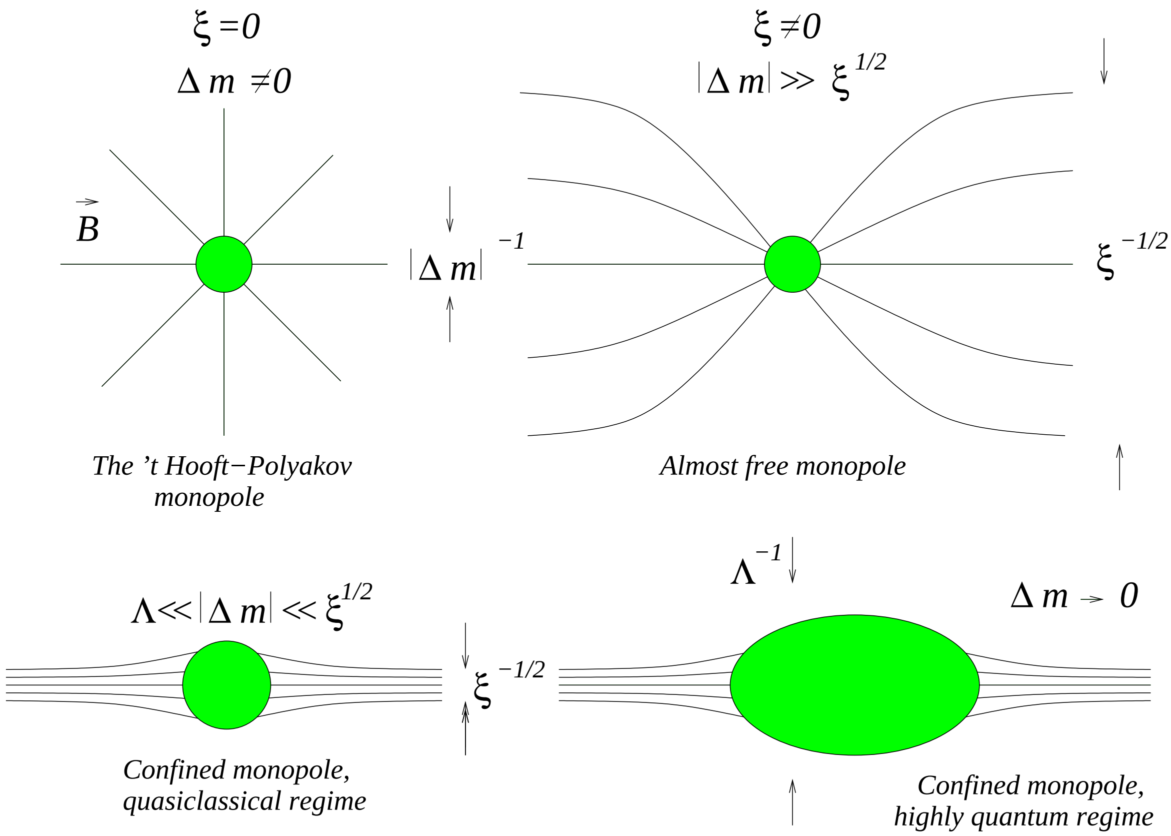


Figure 1.5: Kink-monopoles trapped in the non-Abelian vortex string, first in the Coulomb phase, then with a small FI parameter and for increasing ξ the monopole becomes confined by the vortex string and finally it enters a highly quantum regime. This figure is taken from Ref. [8].

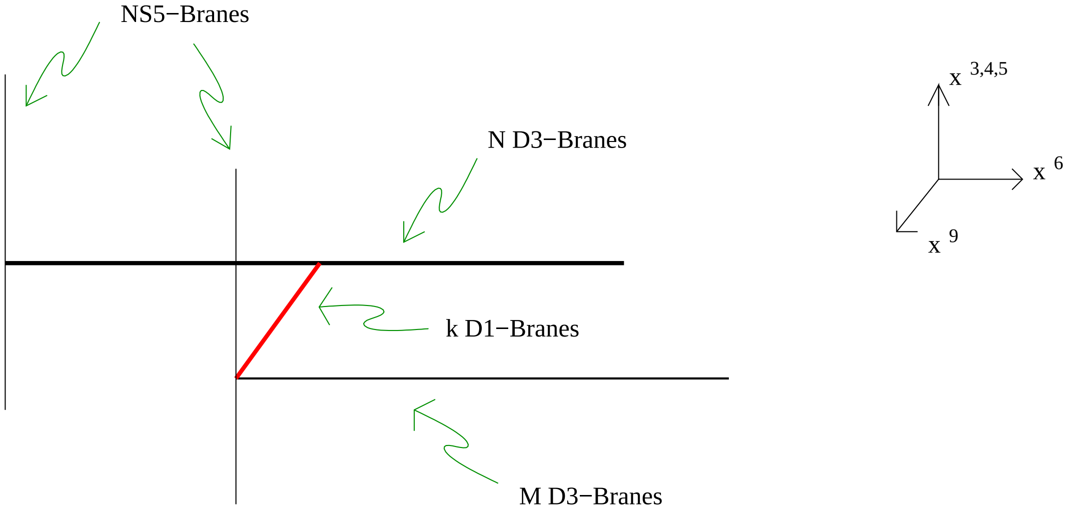


Figure 1.6: A D-brane construction of the vortex theory made by two NS5 branes, N D3 branes and k D-strings. This figure is taken from Ref. [9].

spectrum, excitations of the string can give rise to massless W bosons trapped in the vortex core [106]. This correspondence has been shown to hold both in the weak and strong coupling regime, i.e. an amazing contribution from vortices to non-Abelian gauge theories.

For a single vortex we can understand the effective theory quite well in terms of the orientational vector n. However, for $k > 1$ it proves convenient to consider a D-brane construction to understand the low-energy effective theory. We show a picture of such a construction made by Hanany and Tong [9] in Fig. 1.6. The vortex theory is made by writing down the world-volume theory from the D-strings. This theory is an $\mathcal{N} = (2,2), U(k)$ gauge theory in 2 dimensions. It has a $U(k)$ vector multiplet consisting of a gauge field along with three adjoint scalar fields. And furthermore an adjoint chiral multiplet with a complex scalar field Z. Finally, there are N fundamental chiral multiplets with complex fields ψ which come about from the strings connecting the D-strings with the D3-branes. Roughly speaking, ψ are the orientational zero-modes of the vortex string and the eigenvalues of Z are the positions of the k vortices in the vortex plane. Let us conclude this Section by writing down the Lagrangian of the effective vortex-theory

$$\begin{aligned}\mathcal{L}_{k-\text{vortex}} = \mathrm{Tr}\Bigg[&\frac{1}{2e_{2d}^2}\mathcal{D}_\alpha\phi^r\mathcal{D}^\alpha\phi^r + \mathcal{D}_\alpha Z^\dagger\mathcal{D}^\alpha Z + \mathcal{D}_\alpha\psi\mathcal{D}^\alpha\psi^\dagger - \frac{1}{2e_{2d}^2}\left[\phi^r,\phi^s\right]^2 - |[Z,\phi^r]|^2 \\ &- \psi\psi^\dagger\phi^r\phi^r - \frac{e_{2d}^2}{2}\left(\psi\psi^\dagger - \left[Z,Z^\dagger\right] - \xi\mathbf{1}_k\right)^2\Bigg], \end{aligned} \tag{1.253}$$

where the indices $r,s = 1,2,3$. We will leave the interested reader with the literature for further brane constructions.

Part II

Abelian Chern-Simons vortices

CHAPTER 2

Type III vortex

We study topological Chern-Simons vortices in $2+1$ dimensions. It is shown that in the large magnetic flux limit, topological vortices are well described by a Chern-Simons domain wall compactified on a circle with the symmetric phase inside and the asymmetric phase on the outside. A generic renormalizable potential has two dimensionless parameters that can be varied. Variation of only one of them leads to type I and type II vortices, very similar to the Abrikosov-Nielsen-Olesen (ANO) case. Variation of both the parameters leads to a much richer structure. In particular we have found a new type of vortex, which is type I-like for small flux and then becomes type II-like for larger flux. We will denote it the type III vortex. This results in a stable vortex with vorticity greater than one.

2.1 The large magnetic flux limit

We will now review the domain wall in the Chern-Simons model with magnetic flux [192, 203]. It is possible to add magnetic flux to the domain wall, by switching on a gauge field $A_y(x)$ such that $A_y(-\infty) = -\sqrt{2}f$ and $A_y(+\infty) = 0$. The result is a magnetic flux density equal to f [192]. The wall tension can be written as

$$\begin{aligned} T_{\text{wall+flux}} &= \int_{\mathbb{R}} \left[|\partial_x H|^2 + \frac{\kappa^2 \left(\partial_x A_y\right)^2}{8|H|^2} + \frac{1}{2}A_y^2|H|^2 + \frac{1}{\kappa^2}\left(|H|^2-\xi\right)^2|H|^2 \right] \\ &= \int_{\mathbb{R}} \left[\left|\partial_x H + \frac{1}{\kappa}\left(|H|^2-\xi\right)H\right|^2 - \frac{1}{2\kappa}\partial_x\left(|H|^2-\xi\right)^2 \right] \\ &\quad + \int_{\mathbb{R}} \left[\frac{1}{2}\left|A_y H + \frac{\kappa}{2}\frac{\partial_x A_y}{H^\dagger}\right|^2 - \frac{\kappa}{4}\partial_x\left(A_y^2\right) \right] , \end{aligned} \tag{2.1}$$

which yields the BPS tension and BPS wall solutions

$$T_{\text{wall+flux}} = \frac{\xi^2}{2\kappa} + \frac{\kappa f^2}{2} , \quad H_{\text{wall}}(x) = \frac{\sqrt{\xi}}{\sqrt{1+e^{-m(x-x_0)}}} , \tag{2.2}$$

where the mass parameter is defined as $m \equiv 2\xi/\kappa$.

We can now construct the Chern-Simons wall vortex as follows. We consider the compactification of the wall with flux on a circle of radius R (i.e. along the y-direction). The

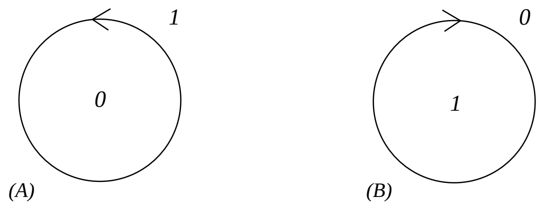


Figure 2.1: Schematic representation of the basic spherical symmetric solitons in the Chern-Simons-Higgs theory. They are both made of the domain wall compactified on a circle and stabilized by the angular momentum. We denote the symmetric vacuum by 0 and the asymmetric vacuum by 1. According to whether the vacuum 0 (unbroken phase) is inside or outside of the circle, we have respectively the topological vortex or the non-topological soliton. In both cases, we have chosen the orientation of the magnetic flux out of the plane (towards the reader).

stabilization is achieved through a balance between the tension of the wall and the energy due to the magnetic field. Notice that the total magnetic flux is the topological constant for the vortex and thus is fixed, while the radius and flux density are related as follows

$$\Phi = 2\pi R f = -2\pi k < 0 \; . \tag{2.3}$$

The energy of the wall vortex as function of the radius is

$$E(R) = \frac{\xi^2 \pi R}{\kappa} + \frac{\kappa \Phi^2}{4\pi R} \; , \tag{2.4}$$

and the minimization of this system gives

$$R = -\frac{\kappa\Phi}{2\pi\xi} = -\frac{\Phi}{\pi m} \; , \qquad E = -\xi\Phi \; . \tag{2.5}$$

This solution saturates the BPS bound (1.177) of the vortex system. This implies that in the large flux limit of the vortex, the solution should exactly become a compactified wall. A useful remark in store, is that the flux density f of the wall vortex (the compactified wall, see Fig. 2.1) is simply proportional to the mass

$$f = -\frac{k}{R} = -\frac{m}{2} \; . \tag{2.6}$$

2.1.1 The BPS vortex

We now want to show that the scalar field of the vortex solution for large winding numbers k, simply has the profile of the domain wall of Eq. (2.2). For the numerical results we shall set the mass $m = 2$, which corresponds to $\kappa = \xi$. The solution for winding number $k = 1$ is shown in Fig. 2.2. We then solve the equations for various values of k. The corresponding profile functions are shown in Fig. 2.3. It is observed that the profile functions of the vortex at large k for the topological vortex are simply described by the profile function of the domain wall (2.2) situated at radius (2.5). The magnetic field is then simply obtained from (1.182).

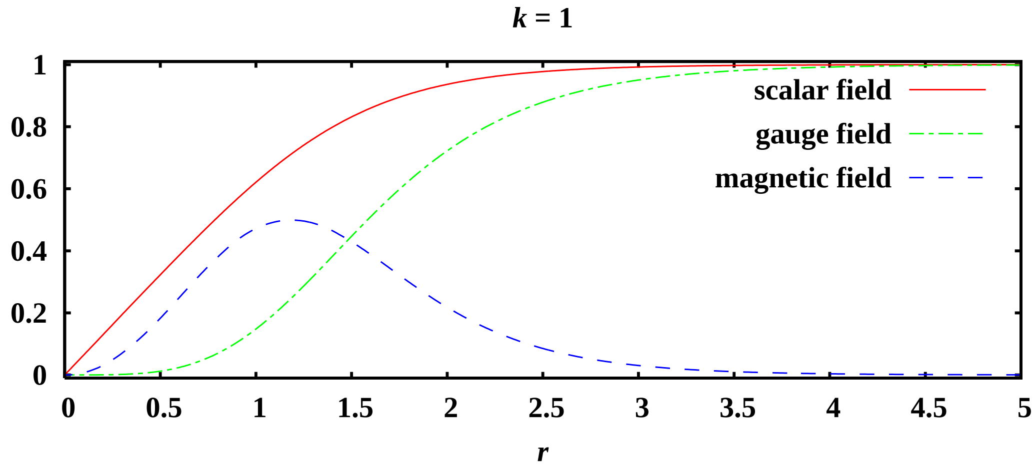


Figure 2.2: Profile functions for the Chern-Simons vortex with $k = 1$, $m = 2$, where the red line (solid) is the scalar field profile, the green line (dash-dotted) is the electromagnetic potential profile (i.e. a given by (1.135)) and the blue line (dashed) is the magnetic field. Notice that the magnetic field is already for $k = 1$ pushed completely outside $r = 0$ and is thus always a ring of flux placed at the vortex boundary.

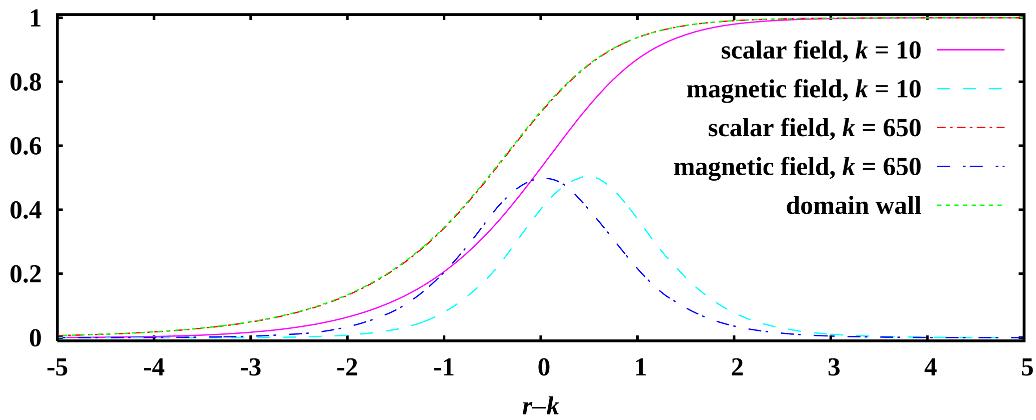


Figure 2.3: Profile functions for Chern-Simons vortex for various k. The radius of the vortex is $R_V = k$ for $m = 2$. Notice that already for $k = 650$, the scalar field and the domain wall coincide.

2.1.2 The non-BPS vortex

A generic renormalizable potential in $2+1$ dimensions is of sixth order, and can be parametrized by two couplings α and β

$$V(|H|) = \frac{\alpha}{\kappa^2}\left(|H|^2 - \xi\right)^2 \left[|H|^2 - \beta\left(|H|^2 - \xi\right)\right] . \tag{2.7}$$

When $\alpha = 1$ and $\beta = 0$ it corresponds to the BPS potential (1.171). We now want to consider the Chern-Simons solitons in the case of a generic potential. We shall here concentrate on the topological vortex, thus in the Higgs phase of the theory.

First we will give some qualitative remarks of what we would expect. Consider the case $\beta = 0$, the potential still has two degenerate vacua; we are only changing the height of the potential. The large flux limit is very useful in order to understand the qualitative behavior. We still expect the vortex to become a compactified wall. The analysis of Section 2.1 should thus go unchanged through, apart from a factor of $\sqrt{\alpha}$ in front of the wall tension. Keeping $\beta = 0$ and varying only α should thus give a phenomenology very similar to that of the Abrikosov-Nielsen-Olesen (ANO) vortex. In the large flux limit the tension is always asymptotically linear in k. Only for $\alpha = 1$ this linear dependence is exact for all values of k. In the intermediate regime the vortices are of type I (attractive) or type II (repulsive) depending on α being smaller or greater than one, respectively.

We now want to consider the case of β different from zero. Since we are focusing on the topological vortex, we choose $\beta < 1$ so that the Higgs phase remains the true vacuum while the Chern-Simons phase acquires a vacuum energy density

$$\varepsilon_0 = \frac{\alpha\beta\xi^3}{\kappa^2} \; , \tag{2.8}$$

and is metastable for $\beta < 1/3$ and an unstable extremum otherwise.

The energy function for the compactified wall now reads

$$E(R) = T_{\mathrm{W}}(\alpha,\beta) 2\pi R + \frac{\kappa\Phi^2}{4\pi R} + \varepsilon_0 \pi R^2 \; , \tag{2.9}$$

where $T_{\mathrm{W}}(\alpha,\beta)$ is the domain wall tension as function of the parameters α, β. Now to obtain the wall vortex, we have to make a minimization with respect to the radius R, in the large k limit. We will see shortly, that R is large in the large k limit, thus we can neglect the first term in (2.9) and the result is

$$R = \sqrt[3]{\frac{\kappa\Phi^2}{8\pi^2\varepsilon_0}} \; , \quad E = \frac{3}{4}\sqrt[3]{\frac{\kappa^2\varepsilon_0\Phi^4}{\pi}} \; , \tag{2.10}$$

which is equivalent to the vortex radius and energy in the large flux limit and furthermore, in terms of k, the radius and energy scale like $R \propto k^{2/3}$ and $E \propto k^{4/3}$. Consistently, the radius goes as $k^{2/3}$, (i.e. the first term in (2.9) goes like R while the two remaining terms go like R^2) and thus our assumption can be justified for sufficiently large k.

The equations of motion for the non-BPS Abelian Chern-Simons vortex are

$$\partial_r^2 a - \frac{1}{r}\partial_r a - \frac{2\left(\partial_r a\right)\left(\partial_r h\right)}{h} + m^2\left(1-a\right)h^4 = 0 \; , \tag{2.11}$$

$$\frac{1}{r}\partial_r\left(r\partial_r h\right) - \frac{k^2}{r^2}\left(1-a\right)^2 h + \frac{k^2\left(\partial_r a\right)^2}{m^2 r^2 h^3} - \frac{1}{2\xi}\frac{\partial V}{\partial h} = 0 \; . \tag{2.12}$$

For a generic potential we obtain

$$\frac{1}{2\xi}\frac{\partial V}{\partial h} = \frac{\alpha}{4}m^2\left(h^2-1\right)\left[3h^2 - 3\beta\left(h^2-1\right) - 1\right]h \; . \tag{2.13}$$

In order to find numerical solutions, we need the boundary conditions at $r \to 0$ and $r \to \infty$. The limiting behavior of the profile functions are for $r \to 0$

$$h = Ar^k \;, \qquad a = Br^{2k+2} \;, \tag{2.14}$$

and for $r \to \infty$

$$h = 1 - Fe^{-\sqrt{\alpha}mr} \;, \qquad a = 1 - Ge^{-mr} \;. \tag{2.15}$$

From this behavior we can form the following conditions

$$\lim_{r\to 0}\,(kh - rh') = 0 \;, \qquad \lim_{r\to 0}\,((2k+2)a - ra') = 0 \;, \tag{2.16}$$

$$\lim_{r\to\infty}\left(h + \frac{h'}{\sqrt{\alpha}m}\right) = 1 \;, \qquad \lim_{r\to\infty}\left(a + \frac{a'}{m}\right) = 1 \;. \tag{2.17}$$

The vortex energy reads

$$E = 2\pi\xi \int dr\, r\left\{\frac{k^2}{m^2}\frac{(\partial_r a)^2}{r^2h^2} + (\partial_r h)^2 + \frac{k^2}{r^2}(1-a)^2 h^2 \right.$$
$$\left. + \frac{\alpha}{4}m^2\left(h^2-1\right)^2\left[h^2 - \beta\left(h^2-1\right)\right]\right\} . \tag{2.18}$$

First, we study numerically the system with $\beta = 0$. The energy function of the compactified wall will, however, change with respect to (2.9) and (2.10) because of zero ε_0 and it is

$$E(R) = \frac{\sqrt{\alpha}\xi^2\pi R}{\kappa} + \frac{\kappa\Phi^2}{4\pi R} \;, \tag{2.19}$$

which gives a radius and energy in the large flux limit of respectively

$$R = \frac{\kappa\Phi}{\sqrt[4]{\alpha}2\pi\xi} \;, \qquad E = -\sqrt[4]{\alpha}\xi\Phi \;. \tag{2.20}$$

In the large flux limit, we can calculate the profile functions for the scalar field and the gauge field analytically, using the $1+1$ dimensional system (the domain wall)

$$\lim_{k\to\infty} h = \frac{1}{\sqrt{1+e^{-\sqrt{\alpha}m(x-x_0)}}} \;, \qquad \lim_{k\to\infty} a = \frac{e^{m(x-x_0)}}{\left[1+e^{\sqrt{\alpha}m(x-x_0)}\right]^{\frac{1}{\sqrt{\alpha}}}} \;. \tag{2.21}$$

We find exact agreement of the numerical integrated profile functions for large values of k with the above results; the vortex becomes a compactified wall in the large k limit.

In Fig. 2.4 is shown the vortex energy normalized for convenience by a numerical factor and $\sqrt[4]{\alpha}$, which puts the different vortex energies on equal footing at large k. We denote by $\mathcal{E} \equiv \frac{E}{k}$ the vortex energy per unit flux. The curves for the vortex energy per unit flux $\mathcal{E}$ approach the large flux limit value (i.e. $2\pi\xi\sqrt[4]{\alpha}$) approximately as $1/k$ (see Fig. 2.4).

We will now turn to the generic case with $\beta \neq 0$. In terms of k, the vortex energy per unit flux $\mathcal{E}$ will go as $k^{\frac{1}{3}}$ in the large flux limit. First a word on our expectations. We seek to combine the type I vortex behavior at small k (attractive force) with the large k behavior

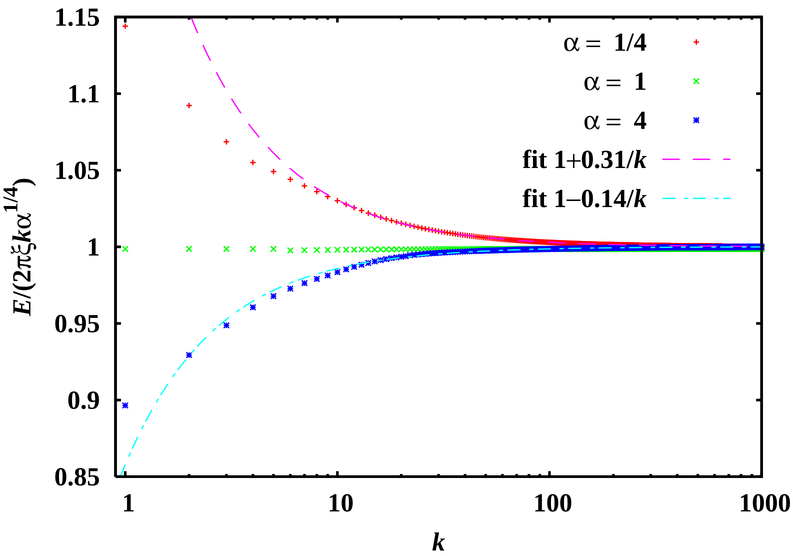


Figure 2.4: Vortex energy divided by $2\pi\xi\sqrt[4]{\alpha}\,k$ as function of the winding number k, for various values of α; $1/4, 1, 4$, corresponding to a type I, a BPS and a type II vortex. $1/k$ fits of the non-BPS vortex energies are shown and are seen to match reasonably at large k.

due to the presence of a non-zero vacuum energy density $\varepsilon_0 = \frac{\alpha\beta\xi^3}{\kappa^2}$. Naïvely, this gives an attractive force for small k and a repulsive force for large k and thus a vortex with finite units of flux greater than one (finite size) as the ones with additional flux will decay.

In Fig. 2.5 is shown the vortex energy per unit of flux for a type I ($\alpha < 1$) vortex with $\alpha = \frac{1}{128}$ where we switch on a small $\beta = 0.03$. For this value of β, the Chern-Simons vacuum is metastable.

From the figure we can define three domains

$$\begin{aligned}
&A : 1 < k \leq k_0 \ , \ \text{where } E(k_0) < E(k) \ , \forall k \neq n_0 \ , \\
&B : k_0 < k < k_1 \ , \ \text{where } k_1 \equiv \left\{ k' \in \mathbb{Z}_+ \,|\, \min\left(\frac{E(k')}{k'}\right) \geq E(1) \right\} \ , \\
&C : k_1 \leq k \ ,
\end{aligned} \tag{2.22}$$

where we have assumed no degeneracy of the lowest energy state. Considering first the domain A, we can prove stability as follows $E(2) < 2E(1)$, is stable; $E(3) < 3E(1)$, and $\frac{E(3)}{3} < \frac{E(2)}{2} \Rightarrow E(3) < E(2) + \frac{E(2)}{2} < E(2) + E(1)$, thus it is stable in all channels.

Generically

$$\begin{aligned}
E(k+l) &< E(k) + \frac{l}{k}E(k) \ , \ \text{for } k + l \leq k_0 \ , \\
&< E(k) + lE(1) \ , \\
&< E(k) + E(l) \ , \quad \text{for} \ , \ l \leq k \ ,
\end{aligned} \tag{2.23}$$

where $k, l \in \mathbb{Z}_+$. Hence, by induction it is seen that the vortices in domain A are stable to decay in any channel.

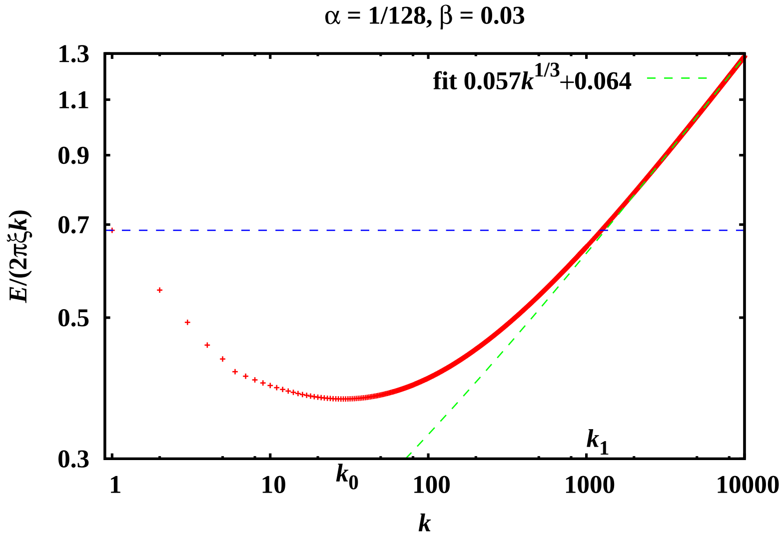


Figure 2.5: Vortex energy divided by $2\pi\xi k$ as function of the winding number k, for a vortex with $\alpha = \frac{1}{128}$ and $\beta = 0.03$. The large k behavior is as predicted proportional to $k^{\frac{1}{3}}$ and the small k behavior is type I-like, thus we have found a vortex with attractive force for small k and repulsive force for large k. We will denote it a type III vortex. k_0 denotes the winding number with the minimal vortex energy.

In domain C it is trivially shown that all vortices are unstable to decay into 1-vortices : $E(k) > kE(1),\ \forall k \geq k_1$.

In domain B it is a priori not so easy to see which channels are allowed and depends on the numerics. In general, the vortices will be unstable to decay in some channels, but it is not certain that there cannot be stable vortices here. We can comment on special points which are unstable, that is, one can easily show that $E(rk_0) > rE(k_0), r \in \mathbb{Z}_+$, however, these windings might be larger than k_1.

The upshot is to note that for small α and β, there will exist a "fat" (winding > 1) vortex with a finite winding number k_0 which is stable and preferred energetically and above a certain winding number larger than k_0 the vortices will decay. This means that vortices will attract to some certain finite size and could be detectable in certain kinds of superconductors and superfluids in 2 dimensional systems. In other words, the vortices are attractive until they reach a critical size and then from that point they will repel additional fluxes. Hence, it is type I at first and when flux adds up, it turns into type II, we could denote this behavior a type III vortex. [1]

We explore now an approximate behavior of the function $k_0(\alpha, \beta)$. In Fig. 2.6, we show the winding number k_0 where the vortex has minimal vortex energy per unit flux (see Fig. 2.5). In the top-most panel is shown k_0 as function of β for fixed $\alpha = 1/128$ and in the

[1] In non-Abelian non-BPS theories there are more possibilities as the forces in general have dependence on the internal properties of such systems. Recently, it was shown [121] that such a non-Abelian vortex in an Extended-Abelian-Higgs theory can have a distance dependent force which turns from attractive to repulsive at some distance.

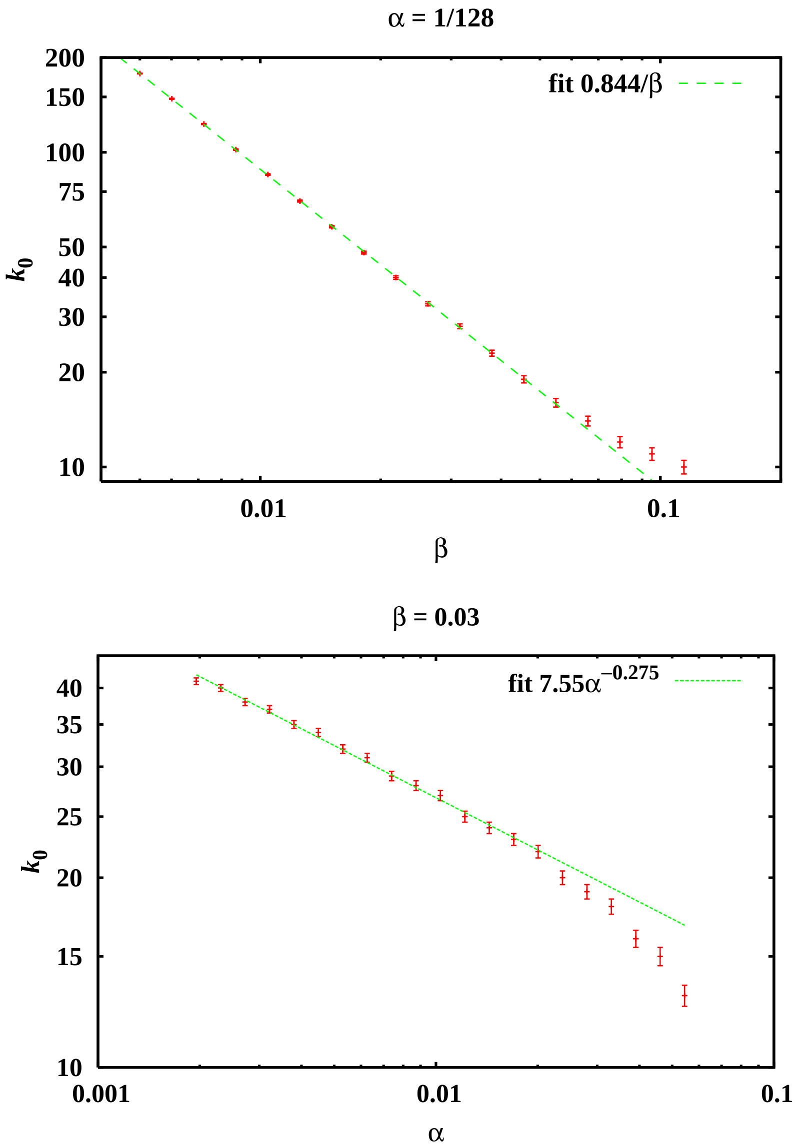


Figure 2.6: The winding number k_0 where the vortex has minimal vortex energy per unit flux $\mathcal{E}$. *Top panel*: k_0 as function of the coupling β for fixed coupling $\alpha = \frac{1}{128}$. For small β ($\beta \lesssim 0.05$), the fit shows that k_0 scales quite well proportional to $\frac{1}{\beta}$. Note that the potential is such that the Chern-Simons vacuum becomes unstable for $\beta \geq \frac{1}{3}$. *Bottom panel*: k_0 as function of the coupling α for fixed coupling $\beta = 0.03$. For small α ($\alpha \lesssim 0.02$), the fit shows good scaling proportional to $\alpha^{-\frac{11}{40}}$. The error-bars are simply a reminder that the function $k_0 \in \mathbb{Z}$.

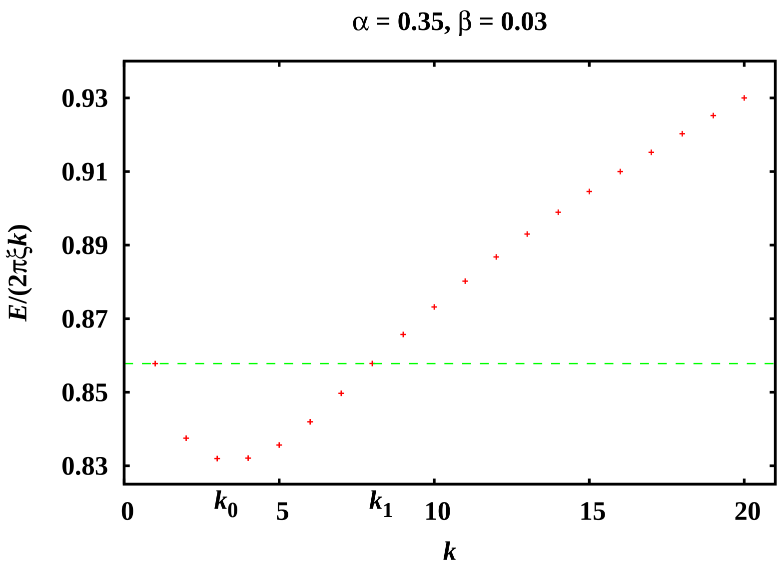


Figure 2.7: Vortex energy divided by $2\pi\xi k$ as function of the winding number k, for a vortex with $\alpha = 0.35$ and $\beta = 0.03$. The couplings are tuned in such away that k_0, the winding with minimal vortex energy per unit flux, is very small (here $k_0 = 3$). Thus the optimal size, energetically, is quite small, but still bigger than the 1-vortex. A simple calculation shows that all vortices with $k > 4$ are unstable to decay.

bottom-most panel, k_0 as function of α for fixed $\beta = 0.03$. In both figures, we have made a fit valid for small values of β, α, respectively.

Around the point $\left(\frac{1}{128}, 0.03\right)$ in (α, β)-space, we can from the fits guess the following approximate formula, which is only valid for small couplings (as the effect is terminated when $\alpha \sim \alpha_{\text{critical}}$ or $\beta \sim \beta_{\text{critical}}$ i.e. k_0 becomes equal to one)

$$k_0 \sim \frac{C}{\alpha^{\frac{11}{40}}\beta}, \tag{2.24}$$

where the constant is $C \sim 0.22$. We have found α_{critical} and β_{critical} to be less than one but of order $\mathcal{O}(10^{-1})$. It could be interesting to see how these functions : $\alpha_{\text{critical}}(\beta), \beta_{\text{critical}}(\alpha)$ behave, but would require a better understanding of the effects kicking in at large couplings.

For phenomenological considerations, it would be interesting to tune k_0 to some small value. As k_0 approaches infinitely large values, the superconductor is effectively of type I and the flux will break the superconducting phase. We expect more than a single point in (α, β)-space to satisfy this condition, actually a line (region) near the critical border : $\alpha(\beta) \lesssim \alpha_{\text{critical}}(\beta)$ or equivalently $\beta(\alpha) \lesssim \beta_{\text{critical}}(\alpha)$. In Fig. 2.7 is shown such a configuration where we have tuned the parameters as : $\alpha = 0.35$ and $\beta = 0.03$. We think that this object could be detectable in two dimensional superconductors in the laboratory, at least in principle.

CHAPTER 3

A large k phase transition

In this Chapter we will study the Maxwell-Chern-Simons theory in order to grasp how the large k limit of the Chern-Simons vortex is smoothly connected with the large k limit of the ANO vortex. First we will construct the domain wall and use it to understand a phase transition at large k in the vortex system.

3.1 Domain wall

Considering a dimensional reduction of the system (1.190), we can obtain the static domain wall energy. We assume that no field depends on x_2 or time x_0 and A_x turns out to vanish in the BPS wall. We thus have

$$
\begin{aligned}
T_{\text{wall}} &= \int_{\mathbb{R}} \left[\frac{1}{2e^2} \left(\partial_x A_0\right)^2 + \frac{1}{2e^2} \left(\partial_x A_2\right)^2 + \frac{1}{2e^2} \left(\partial_x \phi\right)^2 + \frac{1}{2} A_0^2 |H|^2 + |\partial_x H|^2 + \frac{1}{2} A_y^2 |H|^2 \right. \\
&\qquad \left. + \frac{1}{2} \phi^2 |H|^2 + \frac{e^2}{4} \left(|H|^2 - \frac{\kappa}{\sqrt{2}} \phi - \xi \right)^2 \right] , \\
&= \int_{\mathbb{R}} \left[\left| \partial_x H + \frac{1}{\sqrt{2}} \phi H \right|^2 + \frac{1}{2e^2} \left(\partial_x \phi + \frac{e^2}{\sqrt{2}} \left(|H|^2 - \frac{\kappa}{\sqrt{2}} \phi - \xi \right) \right)^2 \right. \\
&\qquad + \frac{1}{2e^2} \left(\partial_x \left(A_0 + A_y\right)\right)^2 + \frac{1}{2} \left| \left(A_0 + A_y\right) H \right|^2 \\
&\qquad \left. - \frac{1}{\sqrt{2}} \partial_x \left(\phi |H|^2 - \frac{\kappa}{2\sqrt{2}} \phi^2 - \xi \phi \right) - \partial_x \left(\frac{1}{e^2} A_y \partial_x A_0 + \frac{\kappa}{4} A_y^2 \right) \right] ,
\end{aligned}
\tag{3.1}
$$

where the Gauss law has been used. We will now consider a flux analogous to the previous case of the Chern-Simons wall, i.e. $A_y(-\infty) = -\sqrt{2} f$ and $A_y(+\infty) = 0$ and furthermore $\partial_x A_0(-\infty) = 0$. The wall tension can be calculated from the boundary term

$$
T_{\text{wall+flux}} = \frac{\xi^2}{2\kappa} + \frac{\kappa f^2}{2} . \tag{3.2}
$$

Notice that the tension coincides exactly with that of the Chern-Simons wall (2.2).

From the tension (3.1) we can read off the BPS equations of motion. First we notice the triviality of the flux equations which simply imply $A_0 = -A_y$ and we have

$$\Xi = \kappa \partial_x \chi \ , \tag{3.3}$$

$$\partial_x^2 \chi - \frac{\kappa e^2}{2} \partial_x \chi - \frac{\xi e^2}{2} (e^{2\chi} - 1) = 0 \ , \tag{3.4}$$

where $\chi \equiv \log(|H|/\sqrt{\xi})$ and $\Xi \equiv -\kappa\phi/\sqrt{2}$. Let us first look at the limits of the wall. Taking $e \to \infty$ we readily obtain the equation for the Chern-Simons wall (2.1). Taking now $\kappa \to 0$ we obtain the system which is the dimensional reduction of the equation for the ANO vortex (1.141).

Note that the flux density is $f = \frac{m}{2}$. Hence, the vorticity is given in the large flux limit and is not a free parameter, as in the case of the uncompactified wall. The numerical results are shown in Fig. 3.1. We set $e^2 = 2$ and start from $\kappa = 1$ and take the limit $\kappa \to 0$. We observe that the magnetic field crawls inside the wall, i.e. to the side of the symmetric phase. This is expected. As κ is sent to zero the VEV of Ξ is pushed away to infinity and the ANO wall vortex emerges. Note that the energy is infinite, unless the wall is compactified on a circle.

3.2 Interpolation between ANO type and CS type

Let us now consider the vortex system (1.199)-(1.201). It will prove convenient to introduce the following dimensionless parameter

$$\eta \equiv \frac{m_\gamma}{m_\kappa} = \frac{\sqrt{\xi} e}{\left(\frac{2\xi}{\kappa}\right)} = \frac{\kappa e}{2\sqrt{\xi}} \ , \tag{3.5}$$

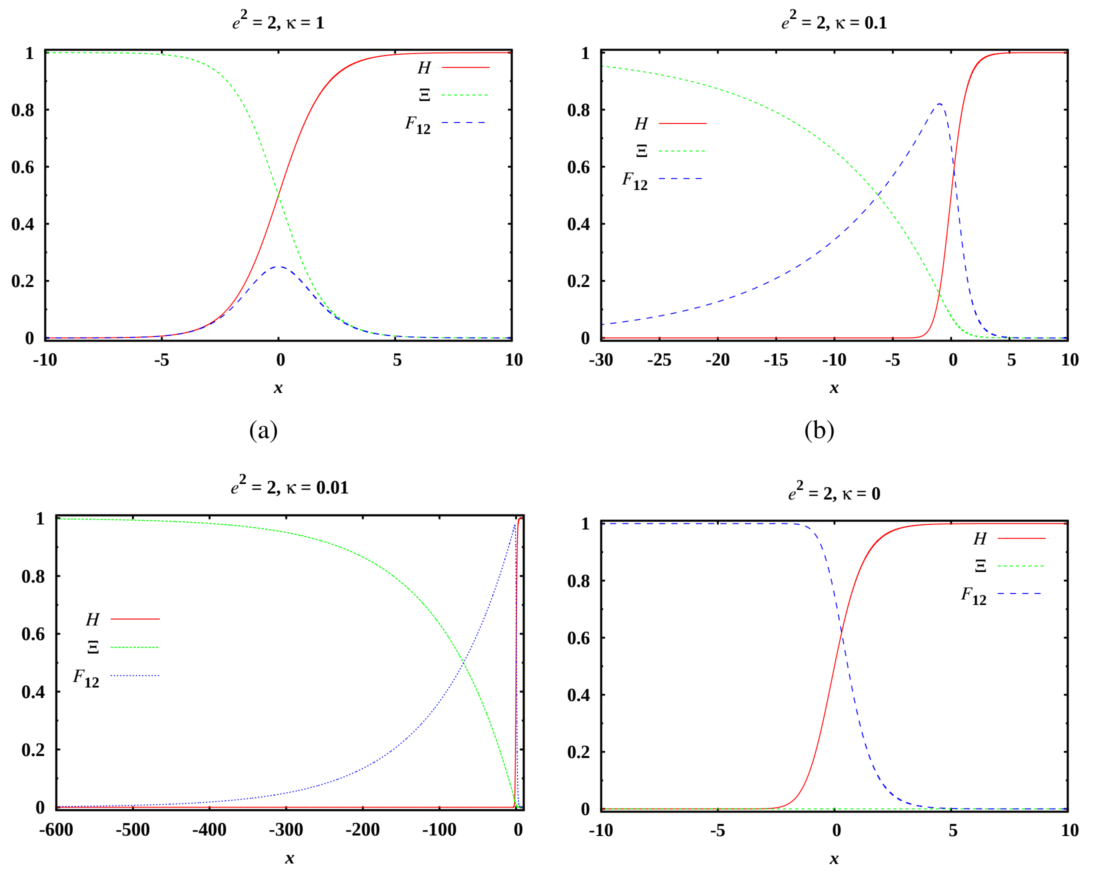
which governs the transition between the pure Maxwell theory ($\eta = 0$) and the pure Chern-Simons theory ($\eta \to \infty$). For all values of η there exists a topological vortex. We already know [192], that in the large k limit the Chern-Simons vortex ($\eta \to \infty$) behaves like a ring of radius

$$R_V^{\mathrm{CS}} = \frac{\kappa}{\xi} k \ . \tag{3.6}$$

The ring is the domain wall that separates the symmetric phase from the Higgs phase. For a generic Maxwell-Chern-Simons vortex, we can use the same argument of [192] to understand the large k limit. As in the pure Chern-Simons case, we still have a symmetric vacuum and a Higgs vacuum. We also have a domain wall between the two vacua and, from the analysis of the previous Section, we know that the wall can support a magnetic flux. We can thus conclude that the large k limit will always be a ring-like structure made of the domain wall. A stabilization calculation using formula (3.2) gives the correct energy of the vortex.

For the pure ANO vortex, the behavior in the large k limit has a completely different nature [171, 172]: it becomes a disc with radius

$$R_V^{\mathrm{ANO}} = \frac{2}{\sqrt{\xi} e} \sqrt{k} \ , \tag{3.7}$$



(a) (b) (c) (d)

Figure 3.1: The fields of the Maxwell-Chern-Simons domain wall with corresponding magnetic field. We set $e = \sqrt{2}$ and in (a) $\kappa = 1$, (b) $\kappa = 0.1$, (c) $\kappa = 0.01$, (d) $\kappa = 0$. Notice how the magnetic field is pushed "inside" the wall vortex. When $\kappa \to 0$ the VEV of Ξ is pushed away to infinity and the Coulomb phase can persist inside the wall vortex.

with the magnetic flux uniformly distributed inside. The masses of the system are

$$m_{\rm CS} = \frac{\kappa e^2}{2} \,, \quad m_\gamma = \sqrt{\xi} e \,, \tag{3.8}$$

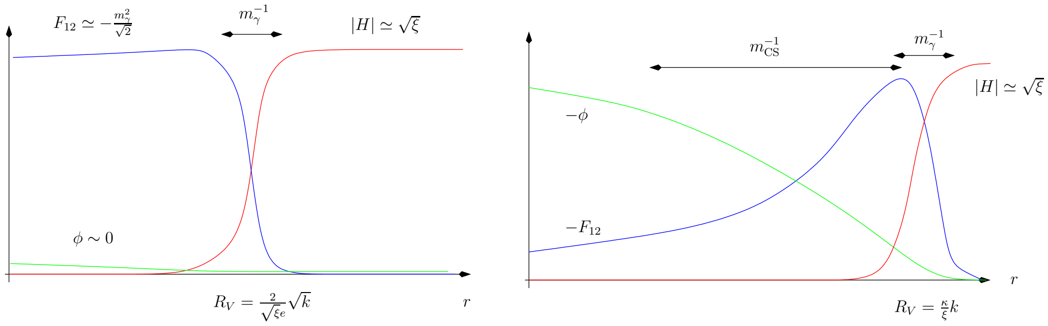
while the approximate profile of the magnetic field and neutral scalar field read

$$F_{12} = -\frac{m_\gamma^2}{\sqrt{2}} e^{m_{\rm CS}(r-R_V)} \,, \quad \phi = \frac{m_\gamma^2}{\sqrt{2} m_{\rm CS}} \left(e^{m_{\rm CS}(r-R_V)} - 1 \right) \,, \tag{3.9}$$

and in this regime inside the wall vortex, we have assumed $H \simeq 0$.

We now want to understand the transition between these two different regimes. Since we can interpolate smoothly between pure Chern-Simons and pure Maxwell theory by changing the parameter η, it must be possible to smoothly interpolate between the disc phase ($R_V \propto \sqrt{k}$) and the ring ($R_V \propto k$) phase. We sketch the two different phases in Fig. 3.2.

To construct the phase diagram (η, k) we have to use some intuition and extrapolate from the previous results of the domain wall with magnetic flux. In the very large k limit,



(a) (b)

Figure 3.2: The profile functions of the vortices in (a) the ANO-disc phase and (b) the Chern-Simons-ring phase. In the ANO phase the magnetic field is a plateau making up the flux tube. The field ϕ is almost zero in this phase. In the Chern-Simons phase the magnetic field will have go to zero exponentially towards the center of the vortex, where the size of the ring is the inverse Chern-Simons photon mass m_{CS}^{-1}.

the vortex always becomes a domain wall-ring. For fixed η, which we will now assume to be so small that the system is in the ANO phase, we will consider the consequence of increasing k. As noted, at some point the system will change into the Chern-Simons phase. When k is sufficiently small, we should compare the Chern-Simons photon mass to the radius of the vortex in the ANO phase

$$R_V^{\mathrm{ANO}} m_{\mathrm{CS}} \sim 1 \;\Rightarrow k_{\mathrm{critical}} \simeq \frac{1}{4\eta^2} \,. \tag{3.10}$$

At some point when the radius starts to scale like the Chern-Simons vortex, i.e. for increasingly larger k, the radius of the Chern-Simons-like vortex should be compared to the Chern-Simons photon mass

$$R_V^{\mathrm{CS}} m_{\mathrm{CS}} \sim 1 \;\Rightarrow k_{\mathrm{critical}} \simeq \frac{1}{2\eta^2} \,. \tag{3.11}$$

As the calculations show, there is a difference of a factor of two, for the two critical winding numbers, so we expect a smooth transition. Now from the above relations we can observe the phase transition happening of course as $\eta \to \infty$ for fixed (small) k, but interestingly it also happens for fixed (small) η as $k \to \infty$. As η becomes just infinitesimally small, k has to be huge to observe this effect. In Fig. 3.3 we show a phase diagram indicating with a shade of gray the phase transition.

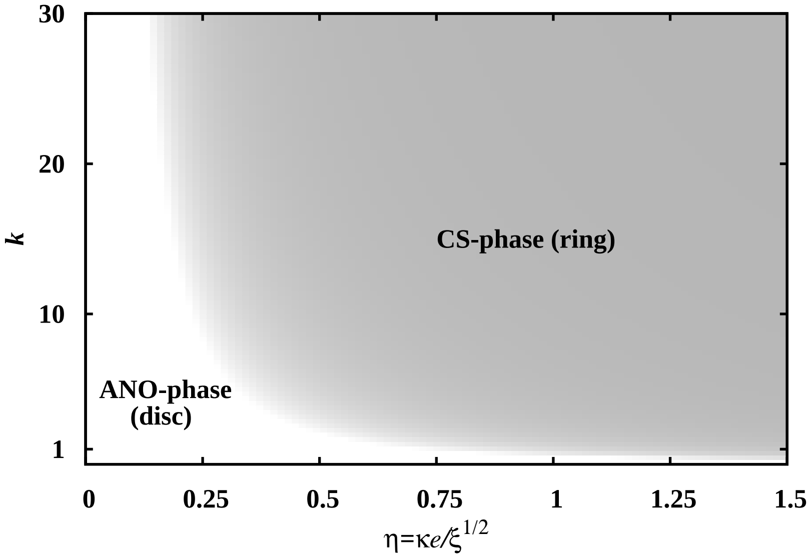


Figure 3.3: A phase diagram for the Maxwell-Chern-Simons vortex. The ANO-type phase and the CS-type phase can be separated by the curves $k = 1/(4\eta^2)$ and $k = 1/(2\eta^2)$. The transition is a smooth transition as illustrated by the tone.

Part III

Vortices and lumps in $SO(N)$ and $USp(2M)$

CHAPTER 4

Special solutions for non-Abelian vortices

In this Chapter we will study special solutions for non-Abelian BPS vortices constructed simply by solving the strong condition by means of an Ansatz. By choosing this Ansatz we can readily characterize solutions in terms of basic group theory and a solution is specified by the $U(1)$ winding number and the coefficients of weight vectors of the dual group $\tilde{G}'$. Interestingly this gives a quantization condition for the vortex solutions formally identical to the well-known one for non-Abelian monopoles due to Goddard-Nuyts-Olive-Weinberg (GNOW).

4.1 GNOW quantization of non-Abelian vortices

Our task is to find all possible moduli matrices which satisfy the weak condition (1.72). In general this is not easy. But certain special solutions can be found readily, and each such solution is characterized by *weight vectors of the dual group*, and is labeled by a set of integers ν_a $(a = 1, \cdots, \mathrm{rank}(G'))$

$$H_0(z) = z^{\nu \mathbf{1}_N + \nu_a \mathcal{H}_a} \in U(1)^{\mathbb{C}} \times G'^{\mathbb{C}} \,, \tag{4.1}$$

where $\nu = k/n_0$ is the $U(1)$ winding number and $\mathcal{H}_a$ are the generators of the Cartan subalgebra of $\mathfrak{g}'$. These special solutions satisfy the strong condition (6.15), given below, with $z_i = 0$. H_0 must be holomorphic in z and *single-valued*, which gives the constraints for a set of integers ν_a

$$(\nu \mathbf{1}_N + \nu_a \mathcal{H}_a)_{ll} \in \mathbb{Z}_{\geq 0} \quad \forall\, l \,. \tag{4.2}$$

Suppose that we now consider scalar fields in an r-representation of G'. The constraint is equivalent to

$$\nu + \nu_a \mu_a^{(i)} \in \mathbb{Z}_{\geq 0} \quad \forall\, i \,, \tag{4.3}$$

where $\vec{\mu}^{(i)} = \mu_a^{(i)}$ $(i = 1, 2, \cdots, \dim(r))$ are the weight vectors for the r-representation of G'. Subtracting pairs of adjacent weight vectors, one arrives at the quantization condition

$$\vec{\nu} \cdot \vec{\alpha} \in \mathbb{Z} \,, \tag{4.4}$$

G'	$\tilde{G}'$
$SU(N)$	$SU(N)/\mathbb{Z}_N$
$U(N)$	$U(N)$
$SO(2M)$	$SO(2M)$
$USp(2M)$	$SO(2M+1)$
$SO(2M+1)$	$USp(2M)$

Table 4.1: Some pairs of dual groups

for every root vector α of G'.

Equation (4.4) is formally identical to the well-known Goddard-Nuyts-Olive-Weinberg (GNOW) quantization condition [204, 205, 206, 207, 208, 209] for the monopoles, and to the vortex flux quantization rule found in Ref. [210]. There is however a crucial difference here, as compared to the case of [204, 205, 206, 207, 208, 209] or [210]. Because of an exact flavor (color-flavor diagonal $G_{\mathrm{C+F}}$) symmetry present here, which is broken by individual vortex solutions, our vortices possess continuous moduli. As will be seen later, at least in the local case these moduli are normalizable, and there are no conceptual problems in their quantization. On the contrary, vortices in Ref. [210] do not have any continuous modulus, while in the case of "non-Abelian monopoles" [204, 205, 206, 207, 208, 209] these interpolating modes suffer from the well-known problems of non-normalizability. Another way the latter difficulty manifests itself is that the naïve "unbroken" group cannot be defined globally due to a topological obstruction [211, 212, 213, 214, 215, 216] in the monopole backgrounds.

The solution of the quantization condition (4.4) is that

$$\tilde{\vec{\mu}} \equiv \vec{\nu}/2 \,, \tag{4.5}$$

is any of the *weight vectors* of the dual group of G'. The dual group, denoted as $\tilde{G}'$, is defined by the dual root vectors [204, 205, 206, 207, 208, 209]

$$\vec{\alpha}^* = \frac{\vec{\alpha}}{\vec{\alpha}\cdot\vec{\alpha}} \,. \tag{4.6}$$

We show examples of dual pairs of groups G', $\tilde{G}'$ in Table 4.1. Note that (4.3) is stronger than (4.4), it has to be zero or a positive integer. This positive quantization condition allows for only a few weight vectors. For concreteness, let us consider scalar fields in the fundamental representation, and choose a basis where the Cartan generators of $G' = SO(2M), SO(2M+1), USp(2M)$ are given by

$$\mathcal{H}_a = \mathrm{diag}\Big(\underbrace{0,\cdots,0}_{a-1},\frac{1}{2},\underbrace{0,\cdots,0}_{M-1},-\frac{1}{2},0,\cdots,0\Big) \,, \tag{4.7}$$

with $a = 1,\cdots,M$. In this basis, special solutions H_0 have the form[1] for $G' = SO(2M)$ and $USp(2M)$

$$H_0^{(\tilde{\mu}_1,\cdots,\tilde{\mu}_M)} = \mathrm{diag}\left(z^{k_1^+},\cdots,z^{k_M^+},z^{k_1^-},\cdots,z^{k_M^-}\right) \,, \tag{4.8}$$

[1] The integers $k_a^\pm$ and k here coincide with $n_a^\pm$ and $n^{(0)}$, respectively, of Ref. [118].

while for $SO(2M+1)$

$$H_0^{(\tilde{\mu}_1,\cdots,\tilde{\mu}_M)} = \mathrm{diag}\left(z^{k_1^+},\cdots,z^{k_M^+},z^{k_1^-},\cdots,z^{k_M^-},z^k\right)\ , \tag{4.9}$$

where $k_a^\pm = \nu \pm \tilde{\mu}_a$.

For example, in the cases of $G' = SO(4), USp(4)$ with a $\nu = 1/2$ vortex, there are four special solutions with $\vec{\tilde{\mu}} = (\frac{1}{2},\frac{1}{2}), (\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2})$

$$\begin{aligned}
H_0^{(\frac{1}{2},\frac{1}{2})} &= \mathrm{diag}(z,z,1,1) = z^{\frac{1}{2}\mathbf{1}_4+1\cdot\mathcal{H}_1+1\cdot\mathcal{H}_2}, &(4.10)\\
H_0^{(\frac{1}{2},-\frac{1}{2})} &= \mathrm{diag}(z,1,1,z) = z^{\frac{1}{2}\mathbf{1}_4+1\cdot\mathcal{H}_1-1\cdot\mathcal{H}_2}, &(4.11)\\
H_0^{(-\frac{1}{2},\frac{1}{2})} &= \mathrm{diag}(1,z,z,1) = z^{\frac{1}{2}\mathbf{1}_4-1\cdot\mathcal{H}_1+1\cdot\mathcal{H}_2}, &(4.12)\\
H_0^{(-\frac{1}{2},-\frac{1}{2})} &= \mathrm{diag}(1,1,z,z) = z^{\frac{1}{2}\mathbf{1}_4-1\cdot\mathcal{H}_1-1\cdot\mathcal{H}_2}. &(4.13)
\end{aligned}$$

These four vectors are the same as the weight vectors of two Weyl spinor representations $\mathbf{2}\oplus\mathbf{2}'$ of $\tilde{G}' = SO(4)$ for $G' = SO(4)$, and the same as those of the Dirac spinor representation $\mathbf{4}$ of $\tilde{G}' = Spin(5)$ for $G' = USp(4)$.

The second example is $G' = SO(5)$ with $\nu = 1$. We have nine special points which are described by $\vec{\tilde{\mu}} = (0,0)$ and $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ and $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$ and thus correspond to

$$\begin{aligned}
H_0^{(0,0)} &= \mathrm{diag}(z,z,z,z,z) = z^{1\cdot\mathbf{1}_5+0\cdot\mathcal{H}_1+0\cdot\mathcal{H}_2}, &(4.14)\\
H_0^{(1,0)} &= \mathrm{diag}(z^2,z,1,z,z) = z^{1\cdot\mathbf{1}_5+2\cdot\mathcal{H}_1+0\cdot\mathcal{H}_2}, &(4.15)\\
H_0^{(0,1)} &= \mathrm{diag}(z,z^2,z,1,z) = z^{1\cdot\mathbf{1}_5+0\cdot\mathcal{H}_1+2\cdot\mathcal{H}_2}, &(4.16)\\
H_0^{(-1,0)} &= \mathrm{diag}(1,z,z^2,z,z) = z^{1\cdot\mathbf{1}_5-2\cdot\mathcal{H}_1+0\cdot\mathcal{H}_2}, &(4.17)\\
H_0^{(0,-1)} &= \mathrm{diag}(z,1,z,z^2,z) = z^{1\cdot\mathbf{1}_5+0\cdot\mathcal{H}_1-2\cdot\mathcal{H}_2}, &(4.18)\\
H_0^{(1,1)} &= \mathrm{diag}(z^2,z^2,1,1,z) = z^{1\cdot\mathbf{1}_5+2\cdot\mathcal{H}_1+2\cdot\mathcal{H}_2}, &(4.19)\\
H_0^{(1,-1)} &= \mathrm{diag}(z^2,1,1,z^2,z) = z^{1\cdot\mathbf{1}_5+2\cdot\mathcal{H}_1-2\cdot\mathcal{H}_2}, &(4.20)\\
H_0^{(-1,1)} &= \mathrm{diag}(1,z^2,z^2,1,z) = z^{1\cdot\mathbf{1}_5-2\cdot\mathcal{H}_1+2\cdot\mathcal{H}_2}, &(4.21)\\
H_0^{(-1,-1)} &= \mathrm{diag}(1,1,z^2,z^2,z) = z^{1\cdot\mathbf{1}_5-2\cdot\mathcal{H}_1-2\cdot\mathcal{H}_2}. &(4.22)
\end{aligned}$$

These nine vectors are the same as the weight vectors of the vector representation $\mathbf{4}$ and the antisymmetric representation $\mathbf{5}$ of the dual group $\tilde{G}' = USp(4)$. The weight vectors corresponding to the $k = 1$ vortex in various gauge groups are given in Fig. 4.1.

4.2 $\mathbb{Z}_2$ parity

As discussed in Ref. [118], the vortices in $G' = SO(N)$ theory are characterized by the first homotopy group

$$\pi_1\left(\frac{SO(N)\times U(1)}{\mathbb{Z}_{n_0}}\right) = \mathbb{Z}\times\mathbb{Z}_2\ , \quad n_0 = \begin{cases} 1 & (N \text{ odd})\ , \\ 2 & (N \text{ even})\ , \end{cases} \tag{4.23}$$

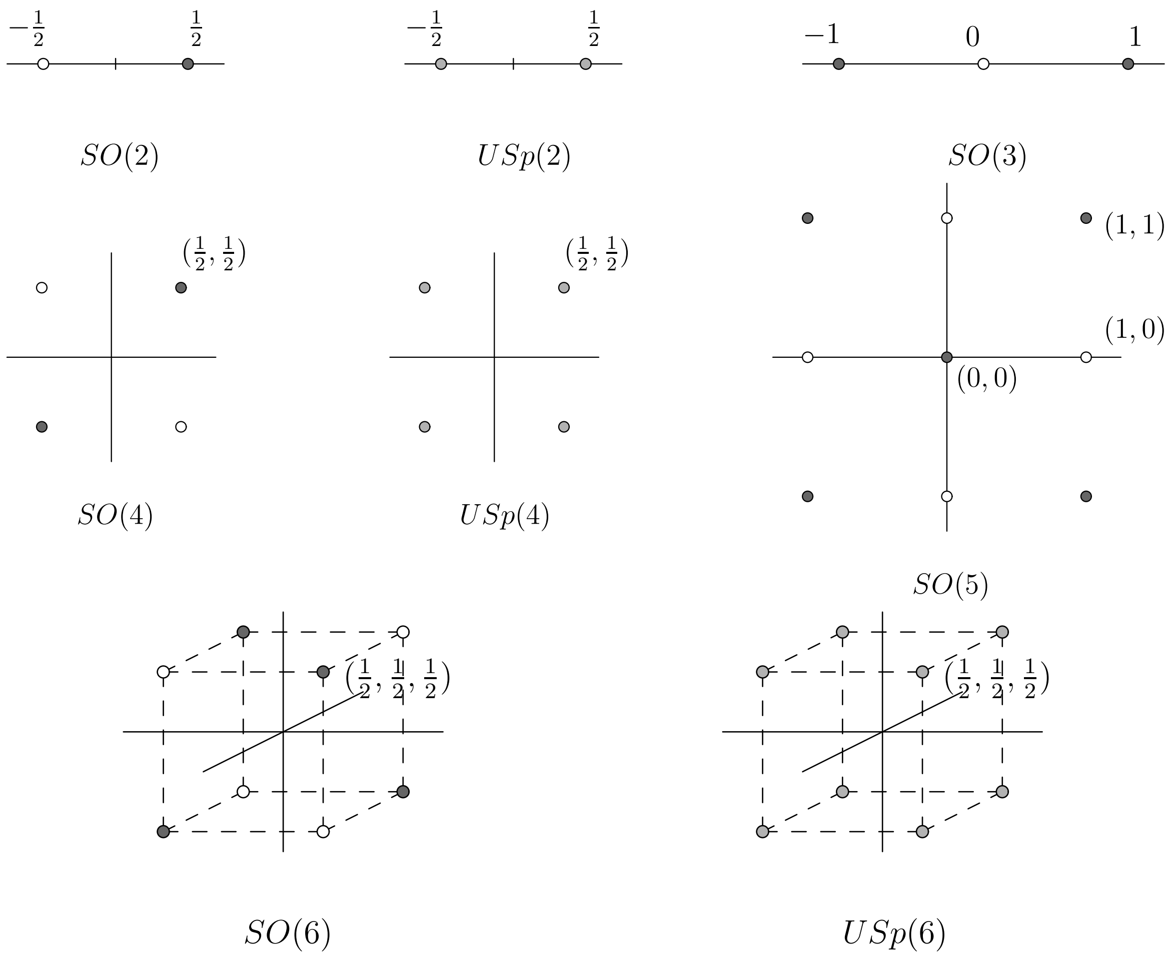


Figure 4.1: The special points for the $k = 1$ vortex.

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$Q_{\mathbb{Z}_2}$
$\frac{1}{2}$	$\frac{1}{2}$	$+1$
$\frac{1}{2}$	$-\frac{1}{2}$	-1
$-\frac{1}{2}$	$\frac{1}{2}$	-1
$-\frac{1}{2}$	$-\frac{1}{2}$	$+1$

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$Q_{\mathbb{Z}_2}$
0	0	$+1$
$\pm\{1$	$0\}$	-1
$\pm\{1$	$\pm1\}$	$+1$

Table 4.2: $k = 1$, $SO(4)$ vortices (left), $k = 1$, $SO(5)$ or $k = 2$, $SO(4)$ (right).

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$\tilde{\mu}_3$	$Q_{\mathbb{Z}_2}$
$\pm\{\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\}$	±1
$\pm\{\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}\}$	∓1

$\tilde{\mu}_1$	$\tilde{\mu}_2$	$\tilde{\mu}_3$	$Q_{\mathbb{Z}_2}$
0	0	0	-1
$\pm\{1$	0	$0\}$	$+1$
$\pm\{1$	1	$0\}$	-1
$\pm\{1$	-1	$0\}$	-1
$\pm\{1$	1	$1\}$	$+1$
$\pm\{-1$	1	$1\}$	$+1$

Table 4.3: $k = 1$, $SO(6)$ cases (left), $k = 1$, $SO(7)$ or $k = 2$ $SO(6)$ (right).

while those of $G' = USp(2M)$ theory correspond to non-trivial elements of

$$\pi_1\left(\frac{USp(2M)\times U(1)}{\mathbb{Z}_2}\right) = \mathbb{Z}\ . \tag{4.24}$$

The vortices in $G' = SO(N)$ carry a $\mathbb{Z}_2$ charge in addition to the usual additive vortex charges. The $\mathbb{Z}_2$ charge can be seen from the dual weight vector $\vec{\tilde{\mu}}$. As a simple example, let us consider the case of $SO(4)$. The dual weight vectors are listed in Table 4.2. Let us compare two states: namely $(\tilde{\mu}_1, \tilde{\mu}_2) = (1/2, 1/2)$ and $(\tilde{\mu}_1, \tilde{\mu}_2) = (1/2, -1/2)$. The difference between them is $\delta(\tilde{\mu}_1, \tilde{\mu}_2) = (0, 1)$: thus one of them can be obtained from the other by a 2π rotation in the (24)-plane in $SO(4)$. As a path from unity to a 2π rotation is a non-contractible loop, they have different $\mathbb{Z}_2$ charges.

On the other hand, the difference between $(\tilde{\mu}_1, \tilde{\mu}_2) = (1/2, 1/2)$ and $(\tilde{\mu}_1, \tilde{\mu}_2) = (-1/2, -1/2)$ is $\delta(\tilde{\mu}_1, \tilde{\mu}_2) = (1, 1)$, hence this is homotopic to the trivial element of $\mathbb{Z}_2$. Therefore, the vortices can be classified by the $\mathbb{Z}_2$-parity, $Q_{\mathbb{Z}_2} = \pm1$. In Figs. 4.1 and 5.3, the dark points correspond to vortices with $Q_{\mathbb{Z}_2} = +1$ while the empty circles correspond to those with $Q_{\mathbb{Z}_2} = -1$.

The $\mathbb{Z}_2$ parity of each special point is defined, in general, as follows:

$$Q_{\mathbb{Z}_2}(k_i^+, k_i^-) = (+)^{\sum_i k_i^+} \times (-)^{\sum_i k_i^-} = (-)^{\sum_i k_i^-}, \tag{4.25}$$

or equivalently in terms of the weight vectors:

$$Q_{\mathbb{Z}_2}\left(H_0^{(\tilde{\mu}_i,\ldots,\tilde{\mu}_M)}\right) = (-)^{\nu M - \sum_i \tilde{\mu}_i}\ . \tag{4.26}$$

CHAPTER 5

Local vortices and orientational moduli

In this Chapter we study local non-Abelian vortices in detail leaving the analyses of semi-local vortices for Chap. 6. The local non-Abelian vortices carry non-Abelian charges under the color-flavor symmetry group. The corresponding moduli parameters are referred to as the internal orientations (or orientational modes) of the vortices. We study the different patches describing the vortex solutions, connectedness properties and finally the transition functions between the various patches. Our study will be systematic for $k = 1, 2$ for $G' = SO(2M), USp(2M)$ and $k = 1$ for $G' = SO(2M + 1)$ (which though has the complexity as $k = 2$ for $G' = SO(2M)$).

5.1 The strong condition

Let us consider a single local vortex. The strong condition is

$$H_0^{\mathrm{T}}(z) J H_0(z) = (z - z_0)^{\frac{2}{n_0}} J \ . \tag{5.1}$$

The parameter z_0 represents the vortex center and is a part of the vortex moduli. Fixing $z_0 = 0$, the solutions to the above condition still possess orientational modes. The moduli space of the orientations should be studied through the solutions to this condition, in principle. However, once a moduli matrix satisfying Eq. (5.1) has been found, other solutions are readily obtained by acting on it with the color-flavor symmetry transformations $G'_{\mathrm{C+F}}$:

$$H_0'(z) \equiv H_0(z) U \ , \quad U \in G'_{\mathrm{C+F}} \ . \tag{5.2}$$

However, $H_0(z)$ is defined only *modulo* V-equivalence, therefore if there exists a V-transformation such that

$$V(z) H_0'(z) = H_0(z) \ , \quad V(z) \in G'^{\mathbb{C}} \ , \tag{5.3}$$

then $H_0'(z)$ and $H_0(z)$ should be regarded as physically the same configuration. Hence, in order to identify the orientational moduli, one needs to identify the flavor rotations which cannot be undone by any V-transformation. In the case of $k = 1$ local vortices with $G' = SO(2M), USp(2M)$, this discussion is sufficient to describe the moduli spaces completely. In the $SO(2M + 1)$ case, and for higher-winding vortices, however, this is not the case. It is here that the moduli matrix formalism shows its power.

5.2 The $k = 1$ local vortex for $G' = SO(2M), USp(2M)$

The strong condition (5.1) with $n_0 = 2$ is satisfied by all special moduli matrices given in Eq. (4.8). For simplicity, let us start with the moduli matrix described by the dual weight vector $\vec{\mu} = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$, i.e.

$$H_0^{(\frac{1}{2},\frac{1}{2},\cdots,\frac{1}{2})}(z) = \mathrm{diag}(\underbrace{z, \cdots, z}_{M}, \underbrace{1, \cdots, 1}_{M}) \,. \tag{5.4}$$

The color-flavor rotation $G'_{\text{C+F}}$ generates other moduli matrices in a $G'_{\text{C+F}}/U(M)$-orbit. It is obvious that the action of the $U(M)$ subgroup of $G' = SO(2M)$ or $G' = USp(2M)$

$$U_0 = \begin{pmatrix} u^{\mathrm{T}} & \\ & u^{-1} \end{pmatrix} \in G'_{\text{C+F}} \,, \quad u \in U(M) \,, \tag{5.5}$$

can be undone by a V-transformation (1.54) due to the fact that

$$H_0^{(\frac{1}{2},\cdots,\frac{1}{2})} U_0 = U_0 H_0^{(\frac{1}{2},\cdots,\frac{1}{2})} \simeq H_0^{(\frac{1}{2},\cdots,\frac{1}{2})} \,. \tag{5.6}$$

Therefore, we find the orientational moduli as parametrizing the following spaces [6]

$$\mathcal{M}_{\text{ori}} = \frac{G'_{\text{C+F}}}{U(M)_{\text{C+F}}} = \frac{SO(2M)}{U(M)} \quad \text{or} \quad \frac{USp(2M)}{U(M)} \,, \tag{5.7}$$

both of which are Hermitian symmetric spaces [217, 160, 218]. The real dimension of the moduli spaces is $M(2M \mp 1) - M^2 + 2 = M(M \mp 1) + 2$. Where the last "+2" correspond to the position of the vortex.

In order to see explicitly $G'_{\text{C+F}}/U(M)$, let us take the following element of G'

$$U = \begin{pmatrix} \mathbf{1}_M & -b^\dagger_{A,S} \\ & \mathbf{1}_M \end{pmatrix} \begin{pmatrix} \sqrt{\mathbf{1}_M + b^\dagger_{A,S} b_{A,S}} & \\ & \left(\sqrt{\mathbf{1}_M + b_{A,S}\, b^\dagger_{A,S}}\right)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_M & \\ b_{A,S} & \mathbf{1}_M \end{pmatrix} \,, \tag{5.8}$$

where b_S (b_A) is an arbitrary M-by-M symmetric (antisymmetric)[1] matrix for the $SO(2M)$ ($USp(2M)$) case. The first two matrices in U can be eliminated by V-transformations, such that the action of U brings the moduli matrix $H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}$ onto the following form

$$\begin{aligned} H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}(z) U \quad \xrightarrow{V} \quad H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}(z; b_{A,S}) &\equiv \begin{pmatrix} z\mathbf{1}_M & \\ b_{A,S} & \mathbf{1}_M \end{pmatrix} \\ &= \begin{pmatrix} z\mathbf{1}_M & \\ & \mathbf{1}_M \end{pmatrix} \begin{pmatrix} \mathbf{1}_M & \\ b_{A,S} & \mathbf{1}_M \end{pmatrix} \,. \end{aligned} \tag{5.9}$$

We denote the patch described by the above moduli matrix the $(\frac{1}{2}, \cdots, \frac{1}{2})$-patch of the manifold $G'_{\text{C+F}}/U(M)$. The complex parameters in the $M \times M$ matrix $b_{A,S}$ are the (local) inhomogeneous coordinates of $\mathcal{M}_{\text{ori}}$. Indeed, the moduli matrix has $\frac{M(M\mp 1)}{2} + 1$ complex

[1] Similar symbols will be used below to indicate a symmetric or antisymmetric constant matrix.

parameters which is in fact the dimension of the moduli space as will be demonstrated in Sec. 6.3. This in turn implies that, in the present case, the moduli space for the local vortex is entirely generated by a G' orbit, except for the position moduli.

By a similar argument we find 2^M patches, starting from the special points $\vec{\tilde{\mu}} = (\pm\frac{1}{2}, \cdots, \pm\frac{1}{2})$ given in Eq. (4.8). Indeed, this can easily be done by means of permutations, e.g.

$$H_0^{(\overbrace{-\frac{1}{2},\cdots,-\frac{1}{2}}^{r},\overbrace{\frac{1}{2},\cdots,\frac{1}{2}}^{M-r})}(z;b_{A,S}) = P_r^{-1} H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}(z;b_{A,S}) P_r \,, \tag{5.10}$$

where the permutation matrix is

$$P_r \equiv \begin{pmatrix} \mathbf{0}_r & & \epsilon\,\mathbf{1}_r & \\ & \mathbf{1}_{M-r} & & \mathbf{0}_{M-r} \\ \mathbf{1}_r & & \mathbf{0}_r & \\ & \mathbf{0}_{M-r} & & \mathbf{1}_{M-r} \end{pmatrix} , \quad P_r^{\mathrm{T}} J P_r = J \,. \tag{5.11}$$

One can easily check that the constraint

$$\left[P_r^{-1} H_0^{(\frac{1}{2},\cdots,\frac{1}{2})} P_r\right]^{\mathrm{T}} J \left[P_r^{-1} H_0^{(\frac{1}{2},\cdots,\frac{1}{2})} P_r\right] = zJ \,,$$

is indeed satisfied. The determinant of the permutation matrices is

$$\det P_r = (-\epsilon)^r \,. \tag{5.12}$$

Note that P_r is an element of G' iff $\det P_r = 1$.

The problem now is to find the transition functions among the 2^M patches just found. As in the case of $U(N)$ vortices [102], the transition functions between the $(\frac{1}{2}, \cdots, \frac{1}{2})$-patch and the $(\underbrace{-\frac{1}{2}, \cdots, -\frac{1}{2}}_{r}, \underbrace{\frac{1}{2}, \cdots, \frac{1}{2}}_{M-r})$-patch are obtained by using the V-transformation (1.54):

$$H_0^{(\overbrace{-\frac{1}{2},\cdots,-\frac{1}{2}}^{r},\overbrace{\frac{1}{2},\cdots,\frac{1}{2}}^{M-r})}(z;b'_{A,S}) = V(z) H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}(z;b_{A,S}) \,. \tag{5.13}$$

By solving the above equation, one obtains the transition functions between the two patches having $\det P_r = 1$ as

$$b'_1 = \epsilon\, b_1^{-1} \,, \quad b'_2 = b_1^{-1} b_2 \,, \quad b'_3 = b_3 + \epsilon\, b_2^{\mathrm{T}} b_1^{-1} b_2 \,, \tag{5.14}$$

where $b_{A,S}$ is decomposed to an r-by-r matrix b_1, an r-by-$(M-r)$ matrix b_2 and an $(M-r)$-by-$(M-r)$ matrix b_3 defined as follows

$$b_{A,S} = \begin{pmatrix} b_1 & b_2 \\ -\epsilon\, b_2^{\mathrm{T}} & b_3 \end{pmatrix} , \quad b_{1,3}^{\mathrm{T}} = -\epsilon\, b_{1,3} \,, \tag{5.15}$$

and similarly for b'_i. The technical details will be postponed till the next Section. This derivation of the quotient space $G'/U(M)$ in the moduli matrix formalism, can be related to the ordinary derivation with $2M$ dimensional vector spaces which we call the orientation vectors. See Sec. 5.3 for the details.

As shown in Eq. (5.12), $\det P_r$ is always $+1$ in the case of $G' = USp(2M)$, while both $+1$ and -1 are possible for $G' = SO(2M)$. Hence, all 2^M patches can be connected for $G' = USp(2M)$. However, two patches which are related by the permutation P_r with $\det P_r = -1$ are disconnected since such a permutation is not an element of $SO(2M)$ but of $O(2M)$ and thus there does not exist any transition function (V-transformation). Therefore, we conclude that the patches for $G' = SO(2M)$ are divided into two disconnected parts according to the sign of $\det P_r = \pm 1$. In summary, the moduli space of the $k = 1$ vortex is

$$\mathcal{M}_{USp(2M)} = \mathbb{C} \times \mathcal{M}^{\rm ori}_{USp(2M)} = \mathbb{C} \times \frac{USp(2M)}{U(M)} \,, \tag{5.16}$$

$$\mathcal{M}_{SO(2M)} = \mathbb{C} \times \mathcal{M}^{\rm ori}_{SO(2M)} = \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_+ \cup \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_- \,, \tag{5.17}$$

with $\mathbb{C}$ being the position moduli. The doubling of the moduli space in the $SO(N)$ case reflects the presence of a $\mathbb{Z}_2$ topological charge for the vortex (see Eq. (4.23)), so that $\mathcal{M}^{\rm ori}_{SO(2M),+} \cap \mathcal{M}^{\rm ori}_{SO(2M),-} = \emptyset$.

Furthermore, the structure of these moduli spaces seems to be consistent with the GNOW duality [204, 205, 206, 207, 208, 209]. The dual of $USp(2M)$ is the $Spin(2M+1)$ group, with a single spinor representation of multiplicity, 2^M. In the case of $SO(2M)$, its GNOW dual is $Spin(2M)$, where the smallest irreducible representations are the two spinor representations of chirality $\pm$, each with multiplicity 2^{M-1}. Actually, the quotient $SO(2M)/U(M)$ is just a space for a pure spinor in $2M$ dimensions [219]. Finally, by embedding the vortex theory into an underlying theory with a larger gauge group which breaks to the group $SO(2M)$ or to $USp(2M)$, what is found here for the vortex moduli and their transformation properties can be translated into the properties of the monopoles appearing at the ends, through the homotopy matching argument [103, 105]. These aspects will be further discussed in a separate article [220].

We have introduced the dual weight diagram $\vec{\tilde{\mu}}$ to represent the special moduli matrices (representative vortex solutions), $H_0^{(\tilde{\mu}_1,\tilde{\mu}_2,\cdots,\tilde{\mu}_M)}(z)$ in Sec. 4. Now we reinterpret them in a slightly different way. The lattice points of the diagram can be thought of as a representation of the patches of the space, where the origin of the local coordinates are just given by these special points. For example, in the case of $G' = SO(2M), USp(2M)$, the lattice point $\vec{\tilde{\mu}} = (\frac{1}{2}, \cdots, \frac{1}{2})$ represents the patch given in Eq. (5.9)[2]. Next we link the lattice points painted with the same color, namely the patches related by the permutation P_r with $\det P_r = +1$. The structure of the moduli space discussed above can easily be read off from the dual weight diagram obtained this way.

The dual lattices formed by special points representatives of connected patches are equal to lattices of irreducible representations of the dual group. On the contrary, two disconnected parts of the moduli space (see $\mathcal{M}_{SO(2M)}$ in Eq. (5.17)) nicely correspond to distinct irreducible representations (two spinor representations of opposite chiralities). In the case

[2]This interpretation gives an intrinsic meaning to the special points . Furthermore, their number is related (in many cases equal) to the Euler character of the moduli space.

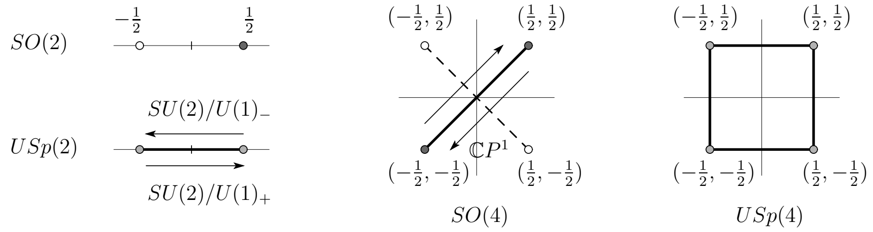


Figure 5.1: The moduli spaces of the $k = 1$ local vortex.

of composite vortices, we will find irreducible representations obtained by tensor compositions of the fundamental ones. This picture holds for all the explicit cases, we could check (low rank groups), and is an important hint of a "semi-classical" emergence of the GNOW duality from the vortex side.

5.2.1 Examples: $G' = SO(2), SO(4), SO(6)$ and $G' = USp(2), USp(4)$

Let us illustrate the structure of the moduli spaces in some simple cases, see Fig. 5.1. The $U(1) \times SO(2) \simeq U(1)_+ \times U(1)_-$ theory has two types of ANO vortices. One type is characterized by $\pi_1(U(1)_+)$ and the other of $\pi_1(U(1)_-)$. They are described by the following moduli matrices

$$H_0^{(\frac{1}{2})} = \begin{pmatrix} z - z_1 & 0 \\ 0 & 1 \end{pmatrix} , \quad H_0^{(-\frac{1}{2})} = \begin{pmatrix} 1 & 0 \\ 0 & z - z_2 \end{pmatrix} . \tag{5.18}$$

Because $USp(2) \simeq SU(2)$, the $G' = USp(2)$ vortex is indeed identical to the $U(2)$ vortex which has been well-studied in the literature. The orientational moduli are $\mathbb{C}P^1 \simeq \frac{SU(2)}{U(1)}$. Note that the special configurations $H_0^{(-\frac{1}{2})} = \mathrm{diag}(1, z)$ and $H_0^{(\frac{1}{2})} = \mathrm{diag}(z, 1)$ are fixed points of the $U(1) \subset SU(2)$ group generated by σ_3: $U(1) = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. One can move from $H_0^{(-\frac{1}{2})}$ to $H_0^{(\frac{1}{2})}$ by using $SU(2)/U(1)$ and vice versa [102]:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}}_{H_0^{(-\frac{1}{2})}} \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_{SU(2)/U(1)} = \underbrace{\begin{pmatrix} 0 & 1/a' \\ -a' & z \end{pmatrix}}_{V\text{-transformation}} \underbrace{\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}}_{H_0^{(\frac{1}{2})}} \underbrace{\begin{pmatrix} 1 & 0 \\ a' & 1 \end{pmatrix}}_{SU(2)/U(1)} , \quad \text{with} \quad aa' = 1 . \tag{5.19}$$

The corresponding dual weight diagram, shown in the bottom-left of Fig. 5.1, represents the fundamental multiplet of the dual $SU(2)$ group. It can be also interpreted as the toric diagram of $\mathbb{C}P^1$.

Next consider $G' = SO(4)$ vortices. We have two different vortices which are characterized by the $\pi_1(SO(4)) = \mathbb{Z}_2$-parity. The orientational moduli again turn out to be

$$\mathbb{C}P^1 \simeq \frac{SO(4)}{U(2)} \simeq \frac{SU(2) \times SU(2)}{U(1) \times SU(2)} \simeq \frac{SU(2)}{U(1)} . \tag{5.20}$$

For instance, we find a similar relation between $H_0^{(-\frac{1}{2},-\frac{1}{2})}$ and $H_0^{(\frac{1}{2},\frac{1}{2})}$

$$\underbrace{\begin{pmatrix} \mathbf{1}_2 & \\ & z\mathbf{1}_2 \end{pmatrix}}_{H_0^{(-\frac{1}{2},-\frac{1}{2})}}\underbrace{\begin{pmatrix} \mathbf{1}_2 & b_A \\ & \mathbf{1}_2 \end{pmatrix}}_{SO(4)/U(2)} = \underbrace{\begin{pmatrix} & {b'_A}^{-1} \\ -b'_A & z\mathbf{1}_2 \end{pmatrix}}_{V\text{-transformation}}\underbrace{\begin{pmatrix} z\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix}}_{H_0^{(\frac{1}{2},\frac{1}{2})}}\underbrace{\begin{pmatrix} \mathbf{1}_2 & \\ b'_A & \mathbf{1}_2 \end{pmatrix}}_{SO(4)/U(2)} , \tag{5.21}$$

with $b_A b'_A = \mathbf{1}_2$. The two special points (the two sites of the dual weight diagram) are again fixed points of the $U(1)$ symmetry, thus the dual weight diagram can be thought of as the toric diagram for $\mathbb{C}P^1$. There are two $\mathbb{C}P^1$'s in this case, see Fig. 5.1. Furthermore, the diagram can alternatively be thought of as representing the reducible $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \oplus (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$ representation of the spinor $Spin(4)$, which is the dual of $SO(4)$.

The diagram for the $G' = USp(4)$ case consists of a single structure where all the 4 points are connected

$$\mathcal{M}^{\text{ori}}_{USp(4)} = \frac{USp(4)}{U(2)} . \tag{5.22}$$

This is consistent with the interpretation of the diagram in Fig. 5.1 as being the weight lattice of the irreducible spinor representation $\mathbf{4}$ of $SO(5)$, which is indeed the GNOW-dual of $USp(4)$ [204, 205, 206, 207, 208, 209].

The last example is $G' = SO(6)$ (see Fig. 5.2). This is another neat example where the orientational moduli are a well-known manifold and its dual weight diagram can be identified with a toric diagram. The orientational moduli space is

$$\mathcal{M}^{\text{ori}}_{SO(6)} = \frac{SO(6)}{U(3)} \simeq \frac{SU(4)}{U(1) \times SU(3)} \simeq \mathbb{C}P^3 . \tag{5.23}$$

The corresponding dual weight diagram is shown in Fig. 5.2. There are two $\mathbb{C}P^3$'s, similarly to the case of $G' = SO(4)$. From the toric diagram, one can easily find the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ subspaces which appear as edges and faces, respectively. Again these two separate parts of the moduli spaces can be interpreted as the two spinor representations, $\mathbf{4} \oplus \mathbf{4}^*$, of opposite chiralities of the dual group

$$Spin(6) \sim SU(4) .$$

5.3 The orientation vectors

We have considered the moduli matrix *per se* and studied the orientational moduli space of the local non-Abelian vortices. Our result for $G' = SO(2M), USp(2M)$ is the quotient space given in Eq. (5.7). These spaces are well-known Hermitian symmetric spaces [217, 160, 218]. They can be embedded in the complex Grassmann space

$$G_{2M,M} \simeq \frac{SU(2M)}{SU(M) \times SU(M) \times U(1)} , \tag{5.24}$$

which is described by a $2M \times M$ complex matrix via a $GL(M, \mathbb{C})$ equivalence relation

$$Gr_{2M,M} \simeq \Phi /\!/ GL(M, \mathbb{C}) = \{\Phi \sim \Phi\mathcal{V}\} , \quad \mathcal{V} \in GL(M, \mathbb{C}) . \tag{5.25}$$

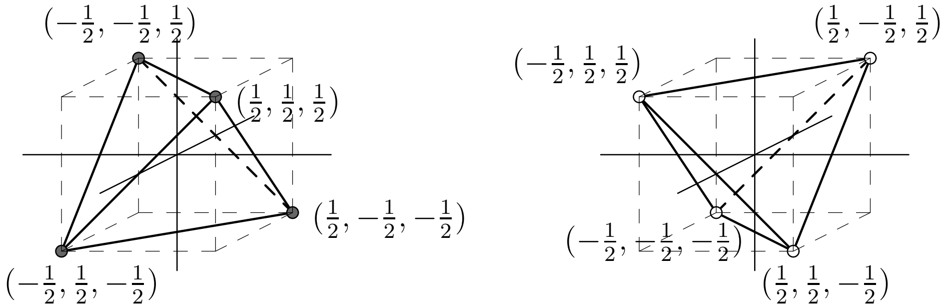


Figure 5.2: The moduli spaces of the $k=1$ local vortex in $G'=SO(6)$.

where the action of $GL(M,\mathbb{C})$ is free. In other words we require the rank of Φ to be M. The embedding is defined by the constraint [160, 218]

$$\Phi^{\mathrm{T}} J \Phi = 0 \, , \tag{5.26}$$

where J is given by Eq. (1.32).

We can relate the matrix Φ to the orientation of the local vortex as follows. Notice that the moduli matrix decreases its rank by M at the "vortex center", $z=z_0$. The orientational moduli can be extracted as M linearly independent $2M$-vectors orthogonal to $H_0(z=z_0)$ [100, 136]

$$H_0(z=z_0)\,\vec{\phi}_i = 0 \, , \qquad (i=1,2,\cdots,M) \, . \tag{5.27}$$

Let us thus define a $2M\times M$ orientational matrix by putting $\vec{\phi}_i$ $(i=1,2,\ldots)$ all together as

$$\Phi = \left(\vec{\phi}_1,\vec{\phi}_2,\cdots,\vec{\phi}_M\right) \, , \qquad H_0(z=z_0)\,\Phi = 0 \, . \tag{5.28}$$

As Φ' given by $\Phi'\equiv\Phi\mathcal{V}$ with $\mathcal{V}\in GL(M,\mathbb{C})$ – which is just a change of the basis – satisfies the same equation (5.27), Φ' represents the same physical configuration as Φ. This leads to the equivalence relation (5.25) and to the complex Grassmannian $Gr_{2M,M}$, as claimed. The isotropic condition (5.26) can be found as follows. The strong condition (5.1) is written as

$$(H_0\Phi)^{\mathrm{T}} J (H_0\Phi) = z\Phi^{\mathrm{T}} J\Phi \, . \tag{5.29}$$

Taking the derivative of this with respect to z, one obtains

$$(\partial H_0\Phi)^{\mathrm{T}} J H_0\Phi + (H_0\Phi)^{\mathrm{T}} J\partial H_0\Phi = \Phi^{\mathrm{T}} J\Phi \, . \tag{5.30}$$

Evaluating this at $z=z_0$ one is led to the constraint (5.26).

The advantage of considering Φ instead of $H_0(z)$ is simplification of the calculation. In the rest of this Subsection, one can completely forget about the previous argument of the moduli matrix. All the results derived from H_0 can be reproduced by Φ alone. Let us illustrate this by taking two examples: $SO(4)$ and $USp(4)$. Then Φ is a 4×2 matrix satisfying $\Phi^{\mathrm{T}} J \Phi = 0$. Since Φ has rank 2, we can generally bring Φ onto the following form by using $GL(2,\mathbb{C})$

$$\Phi^{(\frac{1}{2},\frac{1}{2})}_{SO(4)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -b \\ b & 0 \end{pmatrix} , \quad \Phi^{(\frac{1}{2},\frac{1}{2})}_{USp(4)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ b & c \end{pmatrix} . \tag{5.31}$$

Of course, three other patches $\{\Phi^{(-\frac{1}{2},\frac{1}{2})}, \Phi^{(\frac{1}{2},-\frac{1}{2})}, \Phi^{(-\frac{1}{2},-\frac{1}{2})}\}$ are obtained by fixing $GL(2,\mathbb{C})$ in such a way that the $\{$2-3 rows, 1-4 rows, 3-4 rows$\}$ become the unit matrix, respectively.

The transition functions among them are given through $GL(2,\mathbb{C})$. In the case of $G' = USp(4)$, the transition functions from the $(\frac{1}{2},\frac{1}{2})$-patch to the $\{(-\frac{1}{2},\frac{1}{2}), (\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2})\}$-patches are given by

$$\mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},\frac{1}{2})}_{USp(4)} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}^{-1} , \ \mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(\frac{1}{2},-\frac{1}{2})}_{USp(4)} = \begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix}^{-1} , \ \mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},-\frac{1}{2})}_{USp(4)} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} . \tag{5.32}$$

When the inverse of $\mathcal{V}$ does not exist, such points are not covered by two patches but only by one of them. In the case of $G' = SO(2M)$, neither $\mathcal{V}^{(-\frac{1}{2},\frac{1}{2})\to(\frac{1}{2},\frac{1}{2})}$ nor $\mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},\frac{1}{2})}$ has an inverse. Thus the $(\frac{1}{2},\frac{1}{2})$-patch is disconnected from the $(-\frac{1}{2},\frac{1}{2})$-patch and the $(\frac{1}{2},-\frac{1}{2})$-patch. It connects only with the $(-\frac{1}{2},-\frac{1}{2})$-patch and the transition function is given by

$$\mathcal{V}^{(\frac{1}{2},\frac{1}{2})\to(-\frac{1}{2},-\frac{1}{2})}_{SO(4)} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}^{-1} . \tag{5.33}$$

Similarly, the $(-\frac{1}{2},\frac{1}{2})$-patch and the $(\frac{1}{2},-\frac{1}{2})$-patch are connected. This is a reinterpretation of the $\mathbb{Z}_2$-parity of the local vortex in the model with $G' = SO(4)$, see Fig. 5.1. An extension of this to the local vortex in $G' = SO(2M)$ is straightforward.

5.4 The $k = 2$ local vortex for $G' = SO(2M)$, $USp(2M)$

In the case of $G' = SO(2M), USp(2M)$ theories, the strong condition for the $k = 2$ vortices located at $z = z_1$ and $z = z_2$ is of the form

$$H_0(z)^{\mathrm{T}} J H_0(z) = P(z) J \ , \quad P(z) \equiv (z - z_1)(z - z_2) \ , \tag{5.34}$$

which can equivalently be parametrized as

$$P(z) = (z - z_0)^2 - \delta \ , \quad z_0 = \frac{z_1 + z_2}{2} \ , \quad \delta = \left(\frac{z_1 - z_2}{2}\right)^2 . \tag{5.35}$$

Here z_1 and z_2 stand for the vortex positions which are where the scalar field becomes zero, while z_0 and δ are the center of mass and the relative position (separation) of two vortices, respectively. Several examples of dual weight diagrams are given in Fig. 5.3.

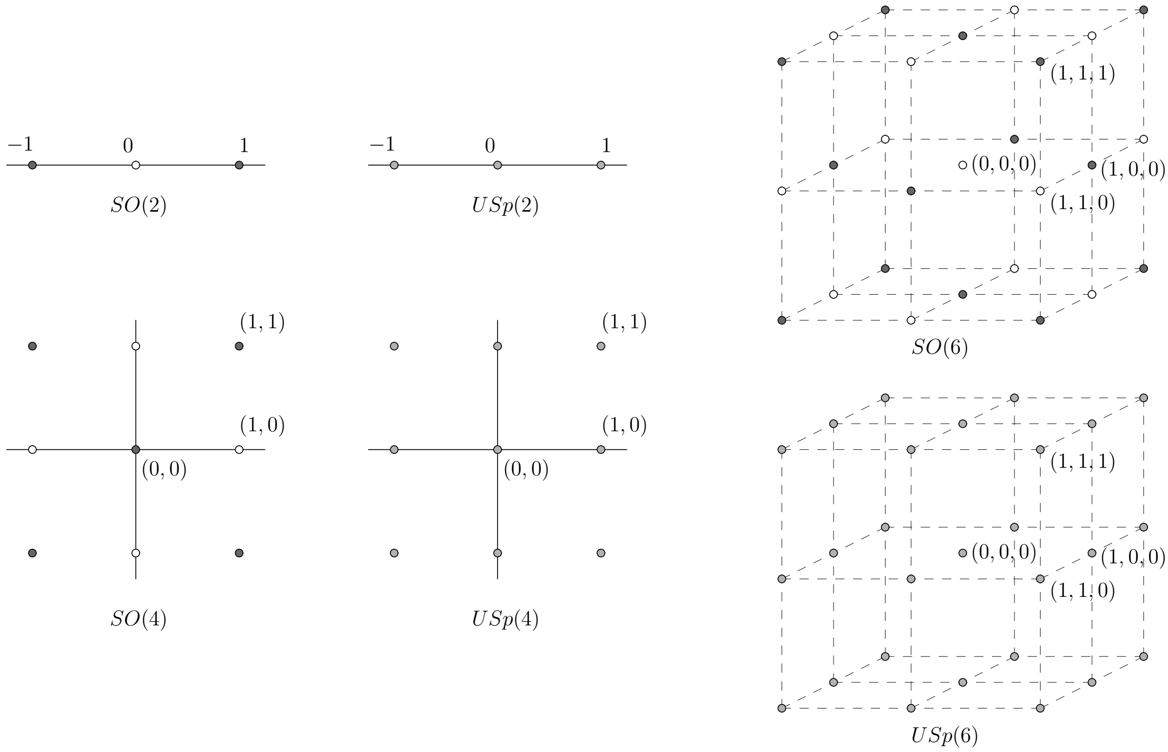


Figure 5.3: The special points for the $k = 2$ vortex.

We will now proceed to the doubly-wound ($k = 2$) vortices in $U(1) \times G'$ gauge theories, with $G' = SO(2M)$ or $USp(2M)$. The $SU(N)_{\mathrm{C+F}}$-orbit structure of the moduli space of k vortices in $U(N)$ gauge theory was studied in Ref. [101] using the Kähler quotient construction of Hanany and Tong [9]. Here we study the orbit structure of the moduli space of $k = 2$ vortices for $G' = SO(2M)$ or $USp(2M)$ more systematically by using the moduli matrix formalism. Before going into the detail, let us recall the properties of the $k = 2$ ANO vortices in the usual Abelian-Higgs model. They can also be studied using the moduli matrix which, in this case is simply a holomorphic function in z, i.e. a second-order polynomial:

$$H_0^{\mathrm{ANO}}(z) = z^2 - \alpha z + \beta = (z - z_1)(z - z_2) \; , \tag{5.36}$$

with $\alpha = z_1 + z_2$ and $\beta = z_1 z_2$. Since these two vortices are indeed identical, we cannot distinguish them. In fact, the moduli matrix is invariant under the exchange of z_1 and z_2. Thus the corresponding moduli space is the symmetric product of $\mathbb{C}$:

$$\mathcal{M}_{\mathrm{ANO}}^{k=2} = \frac{\mathbb{C} \times \mathbb{C}}{\mathfrak{S}_2} \simeq \mathbb{C}^2/\mathbb{Z}_2 \; . \tag{5.37}$$

There is a nice property of the moduli matrix for the local vortices. Suppose H_0^i satisfies the strong condition for k_i local vortices, namely

$$(H_0^i)^{\mathrm{T}} J H_0^i = P_i(z) J \; , \tag{5.38}$$

with a polynomial function of the k_i-th power. Then the product of two matrices $H_0^{(i,j)} \equiv H_0^i H_0^j$ automatically satisfies the strong condition for $k = k_i + k_j$ local vortices:

$$(H_0^{(i,j)})^{\mathrm{T}} J H_0^{(i,j)} = P_i(z) P_j(z) J \,. \tag{5.39}$$

In this way we can construct the moduli matrices for the higher winding number vortices from those with the lower winding numbers, which was found in $U(N)$ vortices [111, 105]. This feature implies that the moduli space for separated local vortices can be constructed as a symmetric product of copies of those of a single local vortex:

$$\mathcal{M}_{\mathrm{sep}}^k \simeq \frac{(\mathbb{C} \times \mathcal{M}_{\mathrm{ori}})^k}{\mathfrak{S}_k} \tag{5.40}$$

The consideration above is valid when the component vortices are separated even for small vortex separations. When two or more vortex axes coalesce, the symmetric product degenerates, and the topological structure of the moduli space undergoes a change. Thus the coincident case must be treated more carefully. We shall study the case of two coincident vortices in detail in the next Section.

Our study of the moduli matrix in the present work is complete up to $k = 2$ vortices ($k = 1$ for odd SO groups). The problem of a complete classification of the moduli matrix for the higher winding number ($k \geq 3$) is left for future work.

The product of moduli matrices, especially for the $G' = SO(N)$ case, gives us a natural understanding in the following sense. The single $G' = SO(N)$ vortex has a $\mathbb{Z}_2$-parity $+1$ or -1. They are physically distinct, hence the $k = 2$ configuration is expected to be classified into three categories by the $\mathbb{Z}_2$-parity of the component vortex as $(Q_{\mathbb{Z}_2}^{(1)}, Q_{\mathbb{Z}_2}^{(2)}) = (+1,+1), (+1,-1), (-1,-1)$. The total $\mathbb{Z}_2$-parity of the configurations with $(Q_{\mathbb{Z}_2}^{(1)}, Q_{\mathbb{Z}_2}^{(2)}) = (+1,+1), (-1,-1)$ is $+1$ while that of $(Q_{\mathbb{Z}_2}^{(1)}, Q_{\mathbb{Z}_2}^{(2)}) = (+1,-1)$ is -1. Therefore, the former and the latter are disconnected. An interesting question is whether $(Q_{\mathbb{Z}_2}^{(1)}, Q_{\mathbb{Z}_2}^{(2)}) = (+1,+1)$ and $(-1,-1)$ are connected or not. The naïve answer would be yes, because the two solutions represent two equivalent objects from the topological point of view. However, the true answer, as we will show, is subtler, and is different for the local and semi-local cases. For the latter case, the two moduli spaces are smoothly connected and in fact are the same space. More interestingly, in the local case they represent two different spaces which intersect at some submanifold. As we shall see, this result is compatible with the interpretation that weight lattices formed by connected special points are in correspondence with irreducible representations of the dual group[3] [118].

The patch structure for the $k = 2$ local vortices in generic $G' = SO(2M)$ or $G' = USp(2M)$ theories is rather complex. In this Subsection, we just present the result without details. The result will be discussed again when we shall consider the generic configurations satisfying the weak condition (1.72) in Sec. 6. The moduli matrix in a generic patch takes

[3] The fact that there is no topology which can explain this disconnection somehow enforces our interpretation in terms of the dual group.

the form

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})}(z) = \begin{pmatrix} P(z)\mathbf{1}_r & 0 & 0 & 0 \\ B_1(z) & (z-z_0)\mathbf{1}_{M-r}+\Gamma_{11} & 0 & \Gamma_{12} \\ A(z) & C_1 & \mathbf{1}_r & C_2 \\ B_2(z) & \Gamma_{21} & 0 & (z-z_0)\mathbf{1}_{M-r}+\Gamma_{22} \end{pmatrix}, \tag{5.41}$$

$$A(z) = a_{1;A,S}\, z + a_{0;A,S} + \lambda_{S,A}\ , \tag{5.42}$$

$$\begin{pmatrix} B_1(z) \\ B_2(z) \end{pmatrix} = -\big((z-z_0)\mathbf{1}_{2(M-r)} + \Gamma\big)\, J_{2(M-r)} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}\ , \tag{5.43}$$

$$\Gamma \equiv \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}\ , \tag{5.44}$$

where $a_{i;A,S}$ $(i = 0, 1)$ is an $r \times r$ constant (anti-)symmetric matrix, C_i is an $r \times (M - r)$ constant matrix and we have defined

$$\lambda_{S,A} \equiv -\frac{1}{2}(C_1,\ C_2)\, J_{2(M-r)} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}\ , \quad J_{2(M-r)} \equiv \begin{pmatrix} & \mathbf{1}_{M-r} \\ \epsilon\,\mathbf{1}_{M-r} & \end{pmatrix}\ . \tag{5.45}$$

The strong condition is now translated into the following form

$$\Gamma^{\mathrm{T}} J_{2(M-r)} + J_{2(M-r)}\Gamma = 0\ , \quad \Gamma^2 = \delta\,\mathbf{1}_{2(M-r)}\ , \qquad (\mathrm{Tr}\,\Gamma = 0)\ . \tag{5.46}$$

Solutions to this condition for separated vortices are discussed in App. B. It is a hard task to study the moduli space collecting all the patches, for generic $SO(2M)$ and $USp(2M)$. A complete analysis of the moduli space in several cases will be given below.

Some of the moduli parameters in Eq. (5.41) are the Nambu-Goldstone (NG) modes associated with global symmetry breaking and the rest are interpreted as so-called quasi-NG modes [221, 222, 223, 224, 225, 226]. The former is, for instance, the overall orientation of the two vortices and the center of mass. The relative separation between two local vortices ($\mathbb{C}$) and some of the relative orientational modes are typical examples of the latter. For two coincident vortices the situation is subtler, but in general there will still be a set of NG modes generated by the $G'_{\mathrm{C+F}}$ symmetry, while the remaining modes are quasi-NG modes. As we will see in the following, the number of the quasi-NG modes is $\left[\frac{M}{2}\right]$ or $\left[\frac{M}{2}\right] - 1$ for $SO(2M)$ and M for $USp(2M)$, which was actually difficult to find without using the moduli matrix formalism.

5.4.1 $G'_{\mathrm{C+F}}$-orbits for coincident vortices

Let us now specialize to the case of the $k = 2$ co-axial (axially symmetric) vortices. The details of the analysis can be found in App. C. Consider a special moduli matrix

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})} = \mathrm{diag}\big(\underbrace{z^2,\cdots,z^2}_{r},\underbrace{z,\cdots,z}_{M-r},\underbrace{1,\cdots,1}_{r},\underbrace{z,\cdots,z}_{M-r}\big)\ . \tag{5.47}$$

Clearly, this vortex breaks the color-flavor symmetry as

$$SO(2M) \to U(r) \times SO(2(M-r)) \ , \quad USp(2M) \to U(r) \times USp(2(M-r)) \ . \tag{5.48}$$

Thus depending on r ($r = 0, 1, \cdots, M$), we have $M + 1$ different $G'_{\text{C+F}}$ orbits. Each orbit reflects the NG modes associated with the symmetry breaking. The different orbits are connected by the quasi-NG modes which are unrelated to symmetry. The total space is stratified with $G'_{\text{C+F}}$-orbits as leaves. To see this, let us consider the following moduli matrix (for $G' = SO(2M)$):

$$H_0 = \begin{pmatrix} z^2\mathbf{1}_{M-2} & & & \\ & z\mathbf{1}_2 & & i\sigma_2\lambda \\ & & \mathbf{1}_{M-2} & \\ & & & z\mathbf{1}_2 \end{pmatrix} = V^{-1} \begin{pmatrix} z^2\mathbf{1}_{M-2} & & & \\ & z^2\mathbf{1}_2 & & \\ & & \mathbf{1}_{M-2} & \\ & -i\sigma_2\lambda^{-1}z & & \mathbf{1}_2 \end{pmatrix} ,$$

$$V = \begin{pmatrix} \mathbf{1}_{M-2} & & & \\ & z\mathbf{1}_2 & & -i\sigma_2\lambda \\ & & \mathbf{1}_{M-2} & \\ & -i\sigma_2\lambda^{-1} & & \mathbf{0}_2 \end{pmatrix} \in SO(2M) \ . \tag{5.49}$$

We can always take λ to be non-negative and real $\mathbb{R}_{>0}$ by means of the color-flavor rotation

$$H_0 \to U^{-1}H_0U \ , \quad U = \begin{pmatrix} \mathbf{1}_{M-2} & & & \\ & a\mathbf{1}_2 & & \\ & & \mathbf{1}_{M-2} & \\ & & & a^{-1}\mathbf{1}_2 \end{pmatrix} \in SO(2M) \ . \tag{5.50}$$

In two limits $\lambda \to 0$ and $\lambda \to \infty$, the moduli matrix (5.49) reduces to the special matrix (5.47) with $r = M - 2$ and $r = M$, respectively. The orbit with intermediate values $0 < \lambda < \infty$ corresponds to the symmetry breaking pattern

$$\frac{SO(2M)}{U(M-2) \times USp(2)} \ . \tag{5.51}$$

In fact, the moduli matrix (5.49) is left invariant under the $USp(2) \in SO(2M)_{\text{C+F}}$ transformations

$$U = \begin{pmatrix} \mathbf{1}_{M-2} & & & \\ & g^{-1} & & \\ & & \mathbf{1}_{M-2} & \\ & & & g^{\mathrm{T}} \end{pmatrix} \in SO(2M) \ , \quad g^{\mathrm{T}}(i\sigma_2)g = i\sigma_2 \ . \tag{5.52}$$

Therefore, the quasi-NG mode λ connects two different $SO(2M)_{\text{C+F}}$ orbits:

$$\frac{SO(2M)}{U(M)} \times \mathbb{Z}_2 \overset{\lambda \to 0}{\longleftarrow} \mathbb{R}_{>0} \times \frac{SO(2M)}{U(M-2) \times USp(2)} \times \mathbb{Z}_2 \overset{\lambda \to \infty}{\longrightarrow} \frac{SO(2M)}{U(M-2) \times SO(4)} \ , \tag{5.53}$$

where the $\mathbb{Z}_2$ factor indicates a permutation, $P^{-1}H_0P$ with $P \in O(2M)/SO(2M)$. This permutation does not belong to the $SO(2M)_{\text{C+F}}$ symmetry, nonetheless it generates a new moduli matrix solution. We thus see, as explained before, how the moduli space of coincident vortices of positive chirality is generically made of two disconnected parts. If $M - r \neq 0$, such a permutation acts trivially or can be pulled back by an $SO(2M)$ rotation on H_0. At these special points the two copies coalesce. Nonetheless we must interpret the two spaces as defining two different composite states of vortices: $(+1, +1)$ and a $(-1, -1)$. This interpretation is fully consistent if one studies interactions in the range of validity of the moduli space approximation [40]. It is easy to realize that, in this approximation, the chirality of each of the component vortices is conserved: two composite states of vortices $(+1, +1)$ and $(-1, -1)$ do not interact, even if their trajectories in the moduli space pass through an intersection submanifold[4]

At the intersection, the dimension of the manifold always reduces by

$$[\dim \mathbb{R}_{>0} - \dim USp(2)] - (-\dim SO(4)) = 4 \, .$$

This can easily be extended to the following moduli matrix, with $t, \alpha \in \mathbb{Z}_{\geq 0}$

$$H_0 = \left(\begin{array}{ccc|ccc} z^2\mathbf{1}_t & & & & & \\ & z^2\mathbf{1}_{2\alpha} & & & & \\ & & z\mathbf{1}_{M-t-2\alpha} & & & \\ \hline 0 & & & \mathbf{1}_t & & \\ & z\tilde{\Lambda} & & & \mathbf{1}_{2\alpha} & \\ & & 0 & & & z\mathbf{1}_{M-t-2\alpha} \end{array}\right) , \tag{5.54}$$

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{\lambda}_1 \tilde{J}_{2\tilde{p}_1} & & \\ & \ddots & \\ & & \tilde{\lambda}_s \tilde{J}_{2\tilde{p}_s} \end{pmatrix} , \tag{5.55}$$

where $\tilde{J}_{2\tilde{p}_i}$ is the invariant tensor of $USp(2\tilde{p}_i)$ and

$$\alpha = \sum_{i=1}^{s} \tilde{p}_i \, , \quad t + 2\alpha \leq M \, , \quad 0 < \tilde{\lambda}_i < \tilde{\lambda}_{i+1} \, . \tag{5.56}$$

An arbitrary patch (5.41) with $\delta = 0$ in the $SO(2M)$ case, can be brought onto the above form as explained in App. C. The set of numbers $(t, s, \tilde{p}_i)$ and the quasi-NG modes λ_i are, of course, independent of r which indicates the patch which we take as a starting point. Note that this is invariant with respect to the group $\prod_{i=1}^{s} USp(2\tilde{p}_i) \in SO(2M)_{\mathrm{C+F}}$

$$U = \text{block-diag}\left(\mathbf{1}_t, g_{2\tilde{p}_1}^{-1}, \cdots, g_{2\tilde{p}_s}^{-1}, \mathbf{1}_{M-t-2\alpha}, \mathbf{1}_t, g_{2\tilde{p}_1}^{\mathrm{T}}, \cdots, g_{2\tilde{p}_s}^{\mathrm{T}}, \mathbf{1}_{M-t-2\alpha}\right) , \tag{5.57}$$

with $g_{2\tilde{p}_i}^{\mathrm{T}} \tilde{J}_{2\tilde{p}_i} g_{2\tilde{p}_i} = \tilde{J}_{2\tilde{p}_i}$. Therefore, the local structure of the $SO(2M)$-orbit has the form

$$\mathbb{R}^s_{>0} \times \frac{O(2M)}{U(t) \times \prod_{i=1}^{s} USp(2\tilde{p}_i) \times O(2u)} , \quad \text{with} \quad t + u + 2\sum_{i=1}^{s} \tilde{p}_i = M \, . \tag{5.58}$$

[4] The question if (or how) these vortices interact beyond the moduli space approximation, and in particular at the quantum level, is an interesting open question.

When we take the limit $\tilde{\lambda}_1 \to 0$, a subgroup $U(t) \times USp(2\tilde{p}_1)$ of the isotropy group gets enhanced to $U(t+2\tilde{p}_1)$ and the orbit shrinks, thus the local structure around the new orbit is given by changing the indices in Eq. (5.58) as follows

$$(s,t,u;\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_s) \overset{\tilde{\lambda}_1\to 0}{\to} (s',t',u';\tilde{p}'_i) = (s-1,t+2\tilde{p}_1,u;\tilde{p}_2,\cdots,\tilde{p}_s)\;. \tag{5.59}$$

In the opposite limit where $\tilde{\lambda}_s \to \infty$, another subgroup $USp(2\tilde{p}_s) \times SO(2u)$ of the isotropy group is getting enlarged to $SO(2u+4\tilde{p}_s)$, hence the local structure around this new orbit is obtained by

$$(s,t,u;\tilde{p}_1,\cdots,\tilde{p}_{s-1},\tilde{p}_s) \overset{\tilde{\lambda}_s\to\infty}{\to} (s'',t'',u'';\tilde{p}''_i) = (s-1,t,u+2\tilde{p}_s;\tilde{p}_1,\cdots,\tilde{p}_{s-1})\;. \tag{5.60}$$

By choosing various $t, \tilde{p}_i$ and taking the limits $\tilde{\lambda}_i \to 0, \infty$, we can reach all the points of the moduli space. However, since these transitions are always induced by the $2\tilde{p}_i \times 2\tilde{p}_i$ matrix $\tilde{J}_{2\tilde{p}_i}$, the patches with only an even number of z^2's in the diagonal element are connected. Analogously, the patches with an odd number of z^2's are mutually connected. Nevertheless, the former and latter remain disconnected and this of course is just a consequence of the different chiralities ($\mathbb{Z}_2$ topological factor).

For instance, by inserting a minimal extension, i.e. the following piece, $\tilde{\lambda}\tilde{J}_2$, the special orbits in Eq. (5.47) can sequentially be shifted as

$$\begin{aligned}\mathrm{diag}(z^2,\cdots,z^2,z^2,z^2,1,\cdots,1,1,1) &\to \mathrm{diag}(z^2,\cdots,z^2,z,z,1,\cdots,1,z,z) \to \cdots\\ &\to \mathrm{diag}(z,\cdots,z,z\cdots,z)\;.\end{aligned}$$

However, the connection pattern depends on whether $SO(2M) = SO(4m)$ or $SO(4m+2)$, see Fig. 5.4. At a generic point ($\tilde{p}_i = 1,\ s = m$) where the color-flavor symmetry is maximally broken the corresponding moduli spaces can locally be written as

$$\mathcal{M}^{k=2,\,\mathrm{ori}}_{SO(4m),+} = \mathbb{R}^m_{>0} \times \frac{SO(4m)}{USp(2)^m} \times \mathbb{Z}_2\;, \tag{5.61}$$

$$\mathcal{M}^{k=2,\,\mathrm{ori}}_{SO(4m),-} = \mathbb{R}^{m-1}_{>0} \times \frac{SO(4m)}{U(1)\times USp(2)^{m-1}\times SO(2)}\;, \tag{5.62}$$

$$\mathcal{M}^{k=2,\,\mathrm{ori}}_{SO(4m+2),+} = \mathbb{R}^m_{>0} \times \frac{SO(4m+2)}{U(1)\times USp(2)^m} \times \mathbb{Z}_2\;, \tag{5.63}$$

$$\mathcal{M}^{k=2,\,\mathrm{ori}}_{SO(4m+2),-} = \mathbb{R}^m_{>0} \times \frac{SO(4m+2)}{USp(2)^m\times SO(2)}\;. \tag{5.64}$$

The two copies of the moduli space, in the case of positive chirality, intersect at some submanifold if $M \neq 1$. The dimensions of these moduli spaces are summarized as

$$\dim_{\mathbb{C}}\left[\mathcal{M}^{k=2,\,\mathrm{ori}}_{SO(2M),\pm}\right] = M^2 - M\;. \tag{5.65}$$

Taking the vortex position into account, the complex dimension of the full moduli space is $M^2 - M + 2$ which is nothing but twice the dimension of the $k = 1$ moduli space.

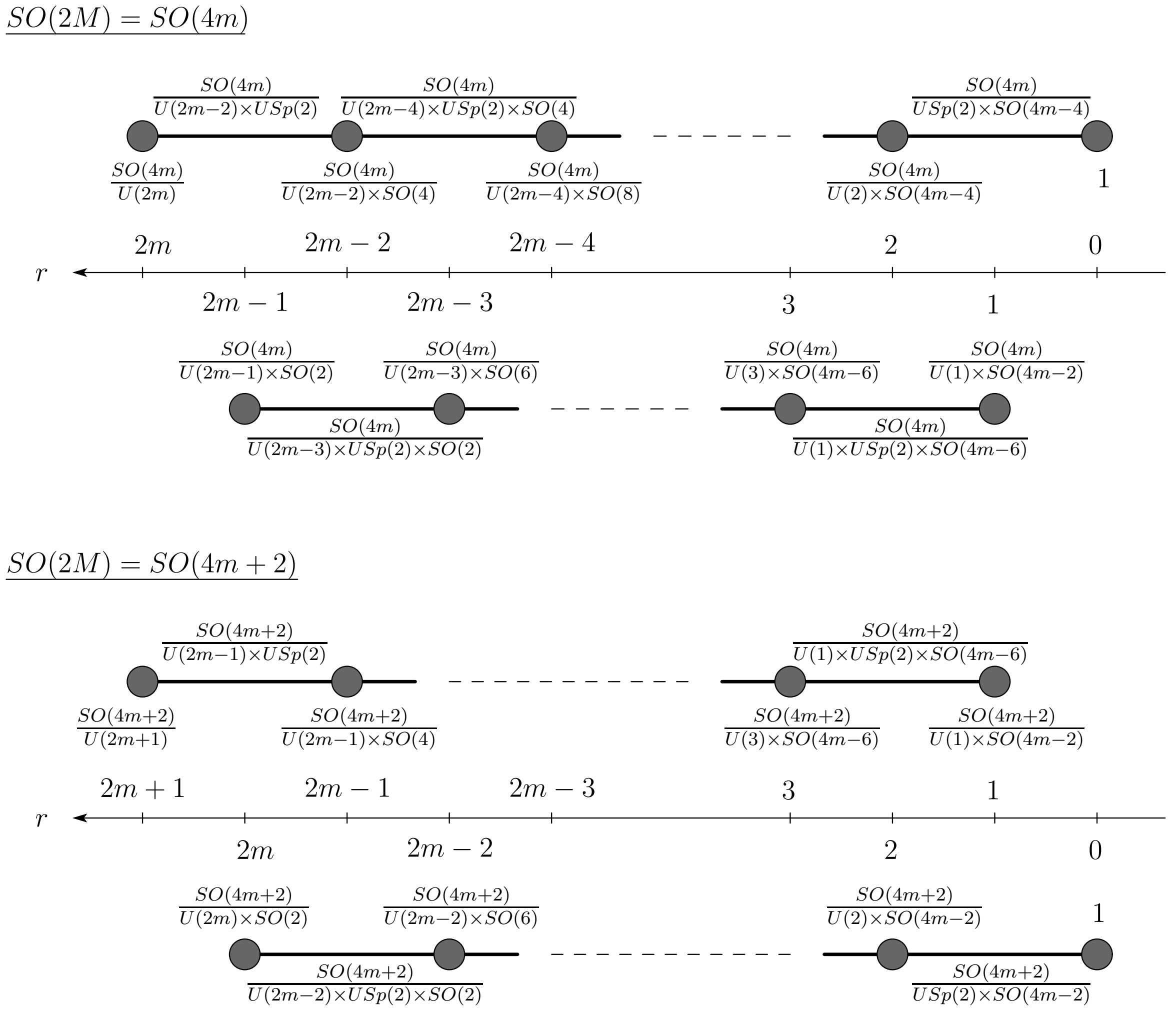


Figure 5.4: Sequences of the $k=2$ vortices in $SO(4m)$ and $SO(4m+2)$. The sites (circles) correspond to the special orbits of Eq. (5.49) and the links connecting them denote the insertion of the minimal pieces $\tilde{\lambda}_i\tilde{J}_2$ such as in Eq. (5.55).

In the case of vortices in $USp(2M)$ theory, we can bring a generic moduli matrix onto the following form

$$H_0 = \left(\begin{array}{ccc|ccc} z^2\mathbf{1}_t & & & & & \\ & z^2\mathbf{1}_\beta & & & & \\ & & z\mathbf{1}_{M-t-\beta} & & & \\ \hline 0 & & & \mathbf{1}_t & & \\ & z\tilde{\Lambda} & & & \mathbf{1}_\beta & \\ & & 0 & & & z\mathbf{1}_{M-t-\beta} \end{array}\right), \tag{5.66}$$

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{\lambda}_1\mathbf{1}_{\tilde{p}_1} & & \\ & \ddots & \\ & & \tilde{\lambda}_s\mathbf{1}_{\tilde{p}_s} \end{pmatrix}, \tag{5.67}$$

with

$$\beta = \sum_{i=1}^{s} \tilde{p}_i, \quad t+\beta \le M, \quad 0 < \tilde{\lambda}_i < \tilde{\lambda}_{i+1} \, . \tag{5.68}$$

This matrix is invariant under $[\prod_{i=1}^{s} O(\tilde{p}_i)] \in USp(2M)$

$$U = \text{block-diag}\left(\mathbf{1}_t, g_{\tilde{p}_1}^{-1}, \cdots, g_{\tilde{p}_s}^{-1}, \mathbf{1}_{M-t-\beta}, \mathbf{1}_t, g_{\tilde{p}_1}^{\mathrm{T}}, \cdots, g_{\tilde{p}_s}^{\mathrm{T}}, \mathbf{1}_{M-t-\beta}\right) \, , \tag{5.69}$$

with $g_{\tilde{p}_i}^{\mathrm{T}} g_{\tilde{p}_i} = \mathbf{1}_{\tilde{p}_i}$. Therefore, the local structure around the $USp(2M)$ orbit is given by

$$\mathbb{R}_{>0}^{s} \times \frac{USp(2M)}{U(t)\times[\prod_{i=1}^{s} O(\tilde{p}_i)]\times USp(2u)} \, , \quad \text{with} \quad t+u+\sum_{i=1}^{s}\tilde{p}_i = M \, . \tag{5.70}$$

In the limit $\tilde{\lambda}_1 \to 0$, the local structures of the orbit changes according to

$$(s,t,u;\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_s) \overset{\tilde{\lambda}_1\to 0}{\to} (s',t',u';\tilde{p}'_i) = (s-1, t+\tilde{p}_1, u; \tilde{p}_2,\cdots,\tilde{p}_s) \, . \tag{5.71}$$

On the other hand, in the opposite limit $\tilde{\lambda}_s \to \infty$, the local structure of the orbit becomes

$$(s,t,u;\tilde{p}_1,\cdots,\tilde{p}_{s-1},\tilde{p}_s) \overset{\tilde{\lambda}_s\to\infty}{\to} (s'',t'',u'';\tilde{p}''_i) = (s-1,t,u+\tilde{p}_s;\tilde{p}_1,\cdots,\tilde{p}_{s-1}) \, . \tag{5.72}$$

Since the minimal insertion is a real positive number $\tilde{\lambda}$, all the special orbits are connected, contrary to the case of the $SO(2M)$ vortices. This is consistent with the fact that there is no $\mathbb{Z}_2$-parity in the $USp(2M)$ case.

At the most generic point where $0 < \tilde{\lambda}_1 < \cdots < \tilde{\lambda}_M$, the color-flavor symmetry is broken down to the discrete subgroup $\mathbb{Z}_2^M$,

$$\mathbb{R}_{>0}^{M} \times \frac{USp(2M)}{\mathbb{Z}_2^M} \, . \tag{5.73}$$

We can read off the dimensions of moduli space for the $k=2$ co-axial local $USp(2M)$ vortices from this

$$\dim_{\mathbb{C}}\left[\mathcal{M}_{USp(2M)}^{k=2,\text{ori}}\right] = \frac{M}{2} + \frac{2M(2M+1)}{4} = M^2+M \, . \tag{5.74}$$

Figure 5.5: The $k = 2$ local vortices for $G' = SO(2), USp(2)$.

5.4.2 Examples: $G' = SO(2), SO(4)$ and $G' = USp(2), USp(4)$

$k = 2$ **local vortices for** $G' = SO(2), USp(2)$

Let us first consider the $G' = SO(2)$ theory. Although there is no $\mathbb{Z}_2$-parity due to the fact that $\pi_1(SO(2)) = \mathbb{Z}$, there are nevertheless two distinct classes of vortices characterized by $\pi_1(U(1)_+)$ and $\pi_1(U(1)_-)$ with $U(1) \times SO(2) \simeq U(1)_+ \times U(1)_-$. Thus there are three possible $k = 2$ configurations. $(\pi_1(U(1)_+), \pi_1(U(1)_-)) = \{(2,0),(0,2),(1,1)\}$, see Fig. 5.5. The corresponding moduli matrices are given by

$$H_0^{(+1)} = \begin{pmatrix} P(z) & 0 \\ 0 & 1 \end{pmatrix}, \quad H_0^{(-1)} = \begin{pmatrix} 1 & 0 \\ 0 & P(z) \end{pmatrix}, \quad H_0^{(0)} = \begin{pmatrix} z - z_1 & 0 \\ 0 & z - z_2 \end{pmatrix}. \tag{5.75}$$

Clearly, z_1 and z_2 are not distinguishable in the first two matrices while they are in the third matrix. This reflects the fact that the configuration consists of two identical vortices and two different vortices, in the two respective cases. Therefore, the moduli space is made of three disconnected pieces

$$\mathcal{M}^{k=2}_{SO(2)} = \mathcal{M}^{(2,0)}_{SO(2)} \cup \mathcal{M}^{(0,2)}_{SO(2)} \cup \mathcal{M}^{(1,1)}_{SO(2)}, \tag{5.76}$$

where these spaces are defined by

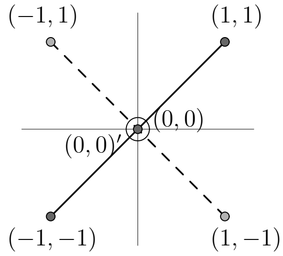
$$\mathcal{M}^{(2,0)}_{SO(2)} = \left(\mathcal{M}^{(1,0)}_{SO(2)} \times \mathcal{M}^{(1,0)}_{SO(2)}\right)/\mathfrak{S}_2 = (\mathbb{C} \times \mathbb{C})/\mathfrak{S}_2 = \mathbb{C}^2/\mathbb{Z}_2, \tag{5.77}$$

$$\mathcal{M}^{(0,2)}_{SO(2)} = \left(\mathcal{M}^{(0,1)}_{SO(2)} \times \mathcal{M}^{(0,1)}_{SO(2)}\right)/\mathfrak{S}_2 = (\mathbb{C} \times \mathbb{C})/\mathfrak{S}_2 = \mathbb{C}^2/\mathbb{Z}_2, \tag{5.78}$$

$$\mathcal{M}^{(1,1)}_{SO(2)} = \mathcal{M}^{(1,0)}_{SO(2)} \times \mathcal{M}^{(0,1)}_{SO(2)} = \mathbb{C}^2. \tag{5.79}$$

The $\mathbb{Z}_2$ factor gives rise to crucial differences in the interactions between these vortices. For instance, a head-on collision of two identical local vortices in $\mathcal{M}^{(2,0)}_{SO(2)}$ or $\mathcal{M}^{(0,2)}_{SO(2)}$ leads to a 90 degree scattering, while such a collision of the two different local vortices living in $\mathcal{M}^{(1,1)}_{SO(2)}$ would be transparent, which yields opposite results for the reconnection of two colliding vortex-strings [112]. Again, this result is a consequence of the fact that vortices with different chiralities must be considered as different, and non-interacting objects.

The next example is $G' = USp(2)$. As was noted earlier the vortices in the $G' = USp(2)$ theory are the ones thoroughly studied due to $USp(2) = SU(2)$. The moduli spaces including the patches and the transition functions for the $k = 2$ vortices, in terms of the moduli matrix, are given in Ref. [111, 105]. We shall not repeat the discussion here. The



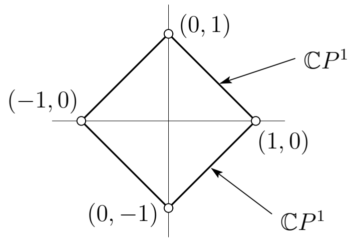


Figure 5.6: The patches of the $k=2$ local vortices in $G' = SO(4)$.

result is [111, 105]

$$\begin{aligned}
\mathcal{M}_{SU(2)}^{k=2,\text{separated}} &\simeq (\mathbb{C}\times\mathbb{C}P^1)^2/\mathfrak{S}_2 ,\\
\mathcal{M}_{SU(2)}^{k=2,\text{coincident}} &\simeq \mathbb{C}\times W\mathbb{C}P^2_{(2,1,1)} \simeq \mathbb{C}\times\mathbb{C}P^2/\mathbb{Z}_2 . \qquad (5.80)
\end{aligned}$$

The dual weight diagram for this case is shown in Fig. 5.5.

$k=2$ local vortices for $G' = SO(4)$

Let us now consider $G' = SO(4)$. As can be seen from Fig. 5.3, there are 9 special points in the entire moduli space. Five out of them have $Q_{\mathbb{Z}_2} = +1$, and the other four have $Q_{\mathbb{Z}_2} = -1$.

Note that the isomorphism $SO(4) \simeq [SU(2)_+ \times SU(2)_-]/\mathbb{Z}_2$ can indeed be complexified as

$$\begin{aligned}
SO(4)^{\mathbb{C}} &\simeq [SL(2,\mathbb{C})_+ \times SL(2,\mathbb{C})_-]/\mathbb{Z}_2 ,\\
[U(1)\times SO(4)]^{\mathbb{C}}/\mathbb{Z}_2 &\simeq [GL(2,\mathbb{C})_+ \times GL(2,\mathbb{C})_-]/\mathbb{C}^* . \qquad (5.81)
\end{aligned}$$

In fact, an arbitrary matrix X satisfying $X^{\mathrm{T}}JX \propto J$ can always be rewritten as

$$\begin{aligned}
&X = \sigma^{-1}(A\otimes B)\sigma = f_+(A)f_-(B) = f_-(B)f_+(A) , \qquad (5.82)\\
&f_+(A) = \sigma^{-1}(A\otimes \mathbf{1}_2)\sigma , \quad f_-(B) = \sigma^{-1}(\mathbf{1}_2\otimes B)\sigma , \quad \sigma = \begin{pmatrix} \mathbf{1}_2 & & \\ & & 1 \\ & -1 & \end{pmatrix} ,
\end{aligned}$$

where $A, B \in GL(2,\mathbb{C})$ and $f_\pm$ define maps from $GL(2,\mathbb{C})_\pm$ to $[U(1)\times SO(4)]^{\mathbb{C}}/\mathbb{Z}_2$. The elements of $GL(2,\mathbb{C})_\pm$, $f_\pm(A)$, are related by the odd parity permutation

$$P^{-1}f_\pm(A)P = f_\mp(A) , \quad P = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix} , \quad (\det P = -1) . \qquad (5.83)$$

Fixed points of this permutation are given by $A \propto \mathbf{1}_2$. This complexified isomorphism tells us that a moduli matrix for $G' = SO(4)$ obeying the strong condition can always be decomposed to a couple of moduli matrices for $G'=SU(2)$ which have been well-studied. This fact simplifies the analysis of the moduli space in the present case. For instance, $f_\pm$ are maps from the moduli matrix for $k=1, G'=SU(2)$ to those of $k=1, SO(4)$ with the parity $Q_{\mathbb{Z}_2}=\pm 1$, since $f_+(\mathrm{diag}(z,1))=\mathrm{diag}(z,z,1,1)$.

Consider first the $Q_{\mathbb{Z}_2}=+1$ patches. There are corresponding patches of the four special points $\vec{\mu}=(\pm 1,\pm 1),(\pm 1,\mp 1)$. The $(1,1)$-patch is explicitly given by the moduli matrix

$$H_0^{(1,1)} \;=\; \begin{pmatrix} z^2+b_1z+b_2 & & & \\ & z^2+b_1z+b_2 & & \\ & -b_3z-b_4 & 1 & \\ b_3z+b_4 & & & 1 \end{pmatrix}, \tag{5.84}$$

with $(z-z_1)(z-z_2)=z^2+b_1z+b_2$. The rest of the patches $H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}$ can be obtained by appropriate permutations of $H_0^{(1,1)}$. Note that the special point $\vec{\mu}=(0,0)$ of the moduli space has two different vicinities which we call the $(0,0)_+$-patch and the $(0,0)_-$-patch, that is, the point $\vec{\mu}=(0,0)$ is on an intersection of two submanifolds. In fact, we find that the two different matrices

$$H_0^{(0,0)+} = \begin{pmatrix} z-a_1 & & & a_4 \\ & z-a_1 & -a_4 & \\ & a_3 & z-a_2 & \\ -a_3 & & & z-a_2 \end{pmatrix}, \tag{5.85}$$

$$H_0^{(0,0)-} = \begin{pmatrix} z-a_1' & a_4' & & \\ -a_3' & z-a_2' & & \\ & & z-a_2' & a_3' \\ & & -a_4' & z-a_1' \end{pmatrix}, \tag{5.86}$$

with

$$(z-z_1)(z-z_2)=(z-a_1)(z-a_2)+a_3a_4=(z-a_1')(z-a_2')+a_3'a_4'\,, \tag{5.87}$$

are connected at the points where $a_3=a_4=a_3'=a_4'=0$ and $a_1=a_2=a_1'=a_2'$ only. As mentioned, these concrete expressions for the patches can be obtained by the maps from those of the $G'=SU(2)$ case as follows

$$H_0^{(0,0)+}=f_+(h^{(1,1)}(a_i))\,,\quad H_0^{(1,1)}=f_+(h^{(2,0)}(b_i))\,,\quad H_0^{(-1,-1)}=f_+(h^{(0,2)}(c_i))\,, \tag{5.88}$$

$$H_0^{(0,0)-}=f_-(h^{(1,1)}(a_i'))\,,\quad H_0^{(1,-1)}=f_-(h^{(2,0)}(b_i'))\,,\quad H_0^{(-1,1)}=f_-(h^{(0,2)}(c_i'))\,, \tag{5.89}$$

where $h^{(*,*)}(a_i)$ are the moduli matrices for $G'=SU(2), k=2$,

$$h^{(1,1)}(a_i)=\begin{pmatrix} z-a_1 & a_4 \\ -a_3 & z-a_2 \end{pmatrix},$$
$$h^{(2,0)}(b_i)=\begin{pmatrix} z^2+b_1z+b_2 & 0 \\ b_3z+b_4 & 1 \end{pmatrix},\quad h^{(0,2)}(c_i)=\begin{pmatrix} 1 & c_3z+c_4 \\ 0 & z^2+c_1z+c_2 \end{pmatrix}. \tag{5.90}$$

The transition functions among these patches are given by the V-transformation (1.54) with $V(z)=f_+(V_+(z))f_-(V_-(z))$ where $V_\pm(z)$ are those of $G'=SU(2)$, i.e. they are exactly the same as in the $SU(2)$ case [100, 136, 111]. Now, connectedness of the patches is manifest since we know the moduli space for $G' = SU(2)$ is indeed simply connected. The three patches in Eq. (5.88) compose a submanifold $\mathcal{M}^{k=2}_{SO(4),++}$ and Eq. (5.89) composes $\mathcal{M}^{k=2}_{SO(4),--}$. The moduli space with $Q_{\mathbb{Z}_2} = +1$, therefore, can be expressed as

$$\mathcal{M}^{k=2}_{SO(4),+} \simeq \mathcal{M}^{k=2}_{SO(4),++} \cup \mathcal{M}^{k=2}_{SO(4),--} , \quad \mathcal{M}^{k=2}_{SO(4),++} \simeq \mathcal{M}^{k=2}_{SO(4),--} \simeq \mathcal{M}^{k=2}_{SU(2)} , \quad (5.91)$$

where $\mathcal{M}^{k=2}_{SU(2)}$ is shown in Eq. (5.80). As we have shown, these two submanifolds intersect at the fixed point of the permutation (5.83) in the $(0,0)_+$-patch and the $(0,0)_-$-patch

$$\mathcal{M}^{k=2}_{SO(4),++} \cap \mathcal{M}^{k=2}_{SO(4),--} = \mathbb{C} , \quad (5.92)$$

where $\mathbb{C}$ describes the position of the two coincident local vortices, $a_1 = a_2 = a_1' = a_2'$. Note that by comparing the right panel of Fig. 5.5 and the left panel of Fig. 5.6 (with a ± 45 degrees rotation), it is easily seen that the $k = 2$, $U(2)$ moduli spaces are embedded in that of the $SO(4)$ theory.

Let us next study the transition functions among the $Q_{\mathbb{Z}_2} = -1$ patches, $(1,0)$-$(0,1)$-$(-1,0)$-$(0,-1)$. The general form of the moduli matrix in the $(1,0)$-patch is:

$$\begin{aligned} H_0^{(1,0)} &= f_+(h^{(1,0)}(z_1,d_1)) f_-(h^{(1,0)}(z_2,d_2)) && (5.93)\\ &= \begin{pmatrix} (z-z_1)(z-z_2) & & & \\ -d_2(z-z_1) & z-z_1 & & \\ -d_1 d_2 & d_1 & 1 & d_2 \\ -d_1(z-z_2) & & & z-z_2 \end{pmatrix} , && (5.94) \end{aligned}$$

while the other three are

$$\begin{aligned} H_0^{(0,1)} &= f_+(h^{(1,0)}(z_1,d_1)) f_-(h^{(0,1)}(z_2,e_2)) , \\ H_0^{(0,-1)} &= f_+(h^{(0,1)}(z_1,e_1)) f_-(h^{(1,0)}(z_2,d_2)) , \\ H_0^{(-1,0)} &= f_+(h^{(0,1)}(z_1,e_1)) f_-(h^{(0,1)}(z_2,e_2)) , \end{aligned} \quad (5.95)$$

where $h^{(1,0)}$ and $h^{(0,1)}$ are the two patches of $\mathcal{M}^{k=1}_{SU(2)} \simeq \mathbb{C} \times \mathbb{C}P^1$,

$$h^{(1,0)}(z_0,b) = \begin{pmatrix} z-z_0 & \\ -b & 1 \end{pmatrix} , \quad h^{(0,1)}(z_0,b') = \begin{pmatrix} 1 & -b' \\ & z-z_0 \end{pmatrix} . \quad (5.96)$$

Hence, we can conclude that the moduli space of the $k = 2$ local vortices with $Q_{\mathbb{Z}_2} = -1$ is

$$\mathcal{M}^{k=2}_{SO(4),-} \simeq (\mathcal{M}^{k=1}_{SU(2)})^2 \simeq \left(\mathbb{C} \times \mathbb{C}P^1\right)^2 . \quad (5.97)$$

This can be also understood from the dual weight diagrams in Figs. 5.1 and 5.6.

The difference between the moduli spaces in Eq. (5.91) and Eq. (5.97) can be understood as follows. Recall that there exist two kinds of minimal vortices in $G' = SO(2M)$ theory, namely one for $SU(2)_+$ with $Q_{\mathbb{Z}_2} = +1$ and another for $SU(2)_-$ with $Q_{\mathbb{Z}_2} = -1$, see Fig. 5.1. We can then choose two vortices with either the same or a different $\mathbb{Z}_2$-parity in composing the $k = 2$ vortex. Two vortices with the same parity can be regarded as physically

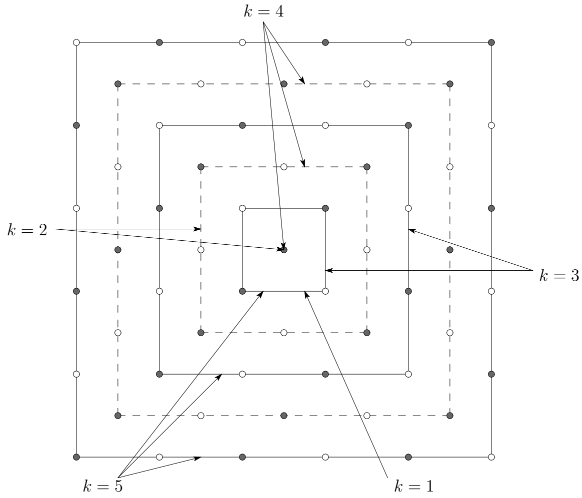


Figure 5.7: The dual weight lattice for $k = 1, 2, 3, 4, 5$ vortices in $G' = SO(4)$.

identical, while those with different parities are distinct. In the case of two identical vortices, the moduli space should be a symmetric product, namely given by Eq. (5.91). Since the total parity $Q^{k=2}_{\mathbb{Z}_2} = +1$ can be made of $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, +1)$ and $(-1, -1)$, one finds two copies, as in Eq. (5.91). In contrast, there is only one possibility for $Q^{k=2}_{\mathbb{Z}_2} = -1$, namely $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (+1, -1)$. The dual weight diagrams are thus quite useful. As a further illustration, let us show the diagrams for some higher-winding vortices with $G' = SO(4)$ in Fig. 5.7, without going into any detail.

$k = 2$ local vortices for $G' = USp(4)$

Consider now the $k = 2$ local vortices for $G' = USp(4)$. Since the moduli for a single ($k = 1$) local vortex requires four parameters, we expect that the $k = 2$ configurations need eight. The moduli matrices including the special points as the origin are of the form

$$H_0^{(0,0)} = (z - z_0)\mathbf{1}_4 + A\,, \tag{5.98}$$

$$H_0^{(1,0)} = \begin{pmatrix} P(z) & 0 & 0 & 0 \\ b_3 b_6 - b_4(z - z_0 + b_5) & z - z_0 + b_5 & 0 & b_6 \\ b_1 z + b_2 & b_3 & 1 & b_4 \\ -b_4 b_7 + b_3(z - z_0 - b_5) & b_7 & 0 & z - z_0 - b_5 \end{pmatrix}, \tag{5.99}$$

with $P(z) = (z - z_0)^2 - \delta = (z - z_0)^2 - (b_5^2 + b_6 b_7)$ and

$$H_0^{(1,1)} = \begin{pmatrix} P(z) & 0 & 0 & 0 \\ 0 & P(z) & 0 & 0 \\ c_3 z + c_4 & c_5 z + c_6 & 1 & 0 \\ c_5 z + c_6 & c_7 z + c_8 & 0 & 1 \end{pmatrix}, \tag{5.100}$$

where $P(z) = z^2 + c_1 z + c_2$. All other patches are connected and can be obtained by suitable permutations. The moduli matrices $H_0^{(1,1)}, H^{(1,0)}$ depend on eight free parameters, as expected. The strong condition is already solved by them, and thus these patches are $\mathbb{C}^8$. The moduli matrix $H_0^{(0,0)}$ has however a more complicated form. The strong condition turns out to be:

$$A^{\mathrm{T}}J + JA = 0\,, \quad A^2 = \delta \mathbf{1}_4\,. \tag{5.101}$$

The first condition tells us that A takes a value in the algebra of $USp(4)$, so

$$A = \begin{pmatrix} -\frac{a_{12}-a_{34}}{2} & a_{35} & a_{13} & a_{15} \\ -a_{45} & -\frac{a_{12}+a_{34}}{2} & a_{15} & -a_{14} \\ a_{24} & a_{25} & \frac{a_{12}-a_{34}}{2} & a_{45} \\ a_{25} & -a_{23} & -a_{35} & \frac{a_{12}+a_{34}}{2} \end{pmatrix}\,. \tag{5.102}$$

Now A has 10 parameters. The second set of constraints comes from imposing the Plücker condition on $a_{ij} = -a_{ji}$ $(i,j = 1,2,3,4,5)$

$$a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk} = 0\,. \tag{5.103}$$

Note that the number of linearly independent conditions is three, hence seven parameters out of ten in the matrix A are linearly independent. Those together with z_0, yield eight degrees of freedom, indeed as expected. In this patch, δ depends on a_{ij} as follows

$$\delta = \frac{1}{4}\left(a_{12} - a_{34}\right)^2 + a_{13}a_{24} - a_{35}a_{45} + a_{15}a_{25}\,. \tag{5.104}$$

Thus the patch $H^{(0,0)}$ is expressed as

$$\begin{aligned} \{H_0^{(0,0)}\} &\simeq \mathbb{C} \times \frac{\{B|B : 2\times 5 \text{ matrix}\}}{SL(2,\mathbb{C})} \simeq \mathbb{C} \times \left(\mathbb{C}^* \rtimes \frac{\{B|B : 2\times 5 \text{ matrix}\}}{GL(2,\mathbb{C})}\right) \\ &\simeq \mathbb{C} \times \left(\mathbb{C} \rtimes \frac{\{B|B : 2\times 5 \text{ matrix of rank } 2\}}{GL(2,\mathbb{C})}\right) = \mathbb{C} \times \left(\mathbb{C} \rtimes Gr_{5,2}\right)\,. \end{aligned} \tag{5.105}$$

The last term in the bracket is a cone whose base space is a $U(1)$ fibration of $Gr_{5,2}$. The tip of this cone corresponds to the origin of the patch, where $a_{ij} = 0$, which is thus a conical singularity in the moduli space. Notice that this is a true singularity of the classical metric on the moduli space. It comes out by applying the strong condition on a smooth set of coordinates [112]. It is an interesting open problem how this singularity affects the interactions of vortices. The transition functions between these patches are easily obtained, for instance, by requiring that $V(z) = H^{(1,1)}(H^{1,0})^{-1}$ be regular with respect to z

$$\begin{aligned} c_1 &= -2z_0\,, & c_2 &= z_0^2 - b_5^2 - b_6 b_7\,, \\ c_3 &= b_1 + \frac{b_4^2}{b_6}\,, & c_4 &= b_2 - \frac{1}{b_6}\left(b_3 b_4 b_6 - b_4^2(b_5 - z_0)\right)\,, \\ c_5 &= -\frac{b_4}{b_6}\,, & c_6 &= b_3 - \frac{b_4}{b_6}(b_5 - z_0)\,, \\ c_7 &= \frac{1}{b_6}\,, & c_8 &= \frac{1}{b_6}(b_5 - z_0)\,. \end{aligned} \tag{5.106}$$

The parameters in $H^{(1,0)}$ are transformed to $a_{ij} = B_{1i}B_{2j} - B_{2i}B_{1j}$ of $H^{(0,0)}$ as

$$B \simeq \frac{1}{\sqrt{b_1}} \begin{pmatrix} 1 & b_3^2 - b_1 b_7 & 0 & -b_2 - z_0 b_1 + b_3 b_4 + b_1 b_5 & -b_3 \\ 0 & -b_2 - z_0 b_1 - b_3 b_4 - b_1 b_5 & 1 & -b_4^2 - b_1 b_6 & b_4 \end{pmatrix}\,. \tag{5.107}$$

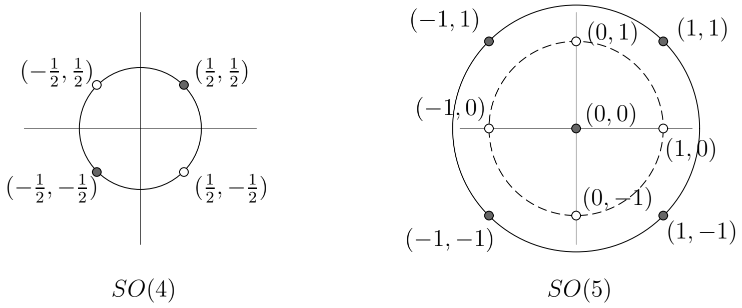


Figure 5.8: Comparison between the single (minimum-winding) vortices in $G' = SO(4)$ and $G' = SO(5)$ theories.

5.5 The $k = 1$ local vortex for $G' = SO(2M + 1)$

Let us now consider the vortex solutions of the $G' = SO(2M + 1)$ theory. The strong condition for the $k = 1$ local vortex positioned at the origin in $G' = SO(2M+1)$ is given by Eq. (5.1) with $n_0 = 1$. It is very similar to the condition Eq. (5.34) for the $k = 2$ coincident vortices ($z_1 = z_2 = 0$) in $G' = SO(2M)$

$$H_0^{\mathrm{T}} J H_0 = z^2 J \,. \tag{5.108}$$

This implies that the complexity of a single local $SO(2M + 1)$ vortex is almost the same as in the case of the $k = 2$ co-axial $SO(2M)$ vortices. Indeed, the corresponding dual weight diagrams, see Figs. 4.1 and 5.3, for instance, are the same.

If however we restrict ourselves to the case of the minimal vortex, there is a startling difference between the case of $SO(2M)$ and that of $SO(2M+1)$. Consider the dual weight diagrams in these two types of theories. In the case of the $SO(2M)$ theory, all the weight vectors have the same length $|\tilde{\vec{\mu}}|^2 = M/4$, whereas those for the $SO(2M+1)$ local vortices have different lengths $|\tilde{\vec{\mu}}|^2$ from 0 to M, see Fig. 5.8 for $SO(4)$ and $SO(5)$. The $M - 1$ dimensional sphere represents an orbit of $G'_{\mathrm{C+F}} = SO(2M)$ or $G'_{\mathrm{C+F}} = SO(2M+1)$ which is nothing but the internal orientation moduli. In the case of $G' = SO(2M)$, the single vortex has only one orbit, hence the moduli space consists of the position $\mathbb{C}$ and the broken color-flavor symmetry $SO(2M)/U(M)$. On the other hand, in the case of $G' = SO(2M + 1)$, there exist multiple orbits corresponding to the NG modes, and furthermore the quasi-NG modes connecting them. For concreteness, let us consider the following moduli matrix

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})}(z) = \mathrm{diag}\big(\underbrace{z^2,\cdots,z^2}_{r},\underbrace{z,\cdots,z}_{M-r},\underbrace{1,\cdots,1}_{r},\underbrace{z,\cdots,z}_{M-r},z\big)\,, \tag{5.109}$$

where r takes on integer values from 0 to M. We now act with the color-flavor symmetry $G'_{\mathrm{C+F}} = SO(2M + 1)$ on the moduli matrix from the right. Hence, the $U(r)$ subgroup in

$SO(2M+1)$ can be absorbed by the V-transformation (1.54):

$$U_0 = \begin{pmatrix} g^{-1} & & & & \\ & \mathbf{1}_{M-r} & & & \\ & & g^{\mathrm{T}} & & \\ & & & \mathbf{1}_{M-r} & \\ & & & & 1 \end{pmatrix} \in U(r) \subset SO(2M+1)\,, \qquad g \in U(r)\,. \tag{5.110}$$

The other subgroup $SO(2(M-r)+1) \subset SO(2M+1)$ can be also absorbed by a V-transformation. Thus the orbit including the special point (5.109) is [118]

$$\mathcal{M}^r_{\text{ori}} = \frac{SO(2M+1)}{U(r) \times SO(2(M-r)+1)}\,. \tag{5.111}$$

The orbit continuously connects the special points corresponding to the dual weight vectors of the same lengths, see Fig. 5.8. Although the internal moduli spaces (5.111) with different r's are not connected by the action of $SO(2M+1)$; these are indeed connected by quasi-NG modes.

The complete moduli space for the $k=1$, $SO(2M+1)$ vortex is very similar to that of $k=2$ co-axial $SO(2M)$ vortices which have been studied in Sec. 5.4.1. A generic solution to the strong condition (5.108) is given by

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})}(z) = \begin{pmatrix} (z-z_0)^2\mathbf{1}_r & 0 & 0 & 0 \\ B_1(z) & (z-z_0)\mathbf{1}_{M-r} + \Gamma_{11} & 0 & \Gamma_{12} \\ A(z) & C_1 & \mathbf{1}_r & C_2 \\ B_2(z) & \Gamma_{21} & 0 & (z-z_0)\mathbf{1}_{M-r+1} + \Gamma_{22} \end{pmatrix}, \tag{5.112}$$

$$\begin{aligned} A(z) &\equiv a_{1;A}\, z + a_{0;A} + \lambda_S\,, & (5.113)\\ \begin{pmatrix} B_1(z) \\ B_2(z) \end{pmatrix} &= -\big((z-z_0)\mathbf{1}_{2(M-r)+1} + \Gamma\big)\, J_{2(M-r)+1} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}\,, & (5.114)\\ \Gamma &\equiv \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}\,, & (5.115) \end{aligned}$$

where $a_{i;A}$ $(i=0,1)$ are $r\times r$ constant anti-symmetric matrices, C_1 is an $r\times(M-r)$ constant matrix, C_2 is an $r\times(M-r+1)$ constant matrix, and we have defined

$$\lambda_S \equiv -\frac{1}{2}(C_1,\, C_2)\, J_{2(M-r)+1} \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix}\,, \quad J_{2(M-r)+1} \equiv \begin{pmatrix} & \mathbf{1}_{M-r} & \\ \mathbf{1}_{M-r} & & \\ & & 1 \end{pmatrix}\,. \tag{5.116}$$

The strong condition is now translated into the following form

$$\Gamma^{\mathrm{T}} J_{2(M-r)+1} + J_{2(M-r)+1}\Gamma = 0\,, \quad \Gamma^2 = 0\,. \tag{5.117}$$

All moduli parameters are included in $a_{i;A}, C_i, \Gamma$. As in the case of $k=2$ co-axial $G' = SO(2M)$ vortices (see App. **C**), $a_{0;A}$ and C_i can be removed by an appropriate color-flavor rotation and Γ satisfying the strong condition (5.117) can be written as (up to $SO(2M+1)_{\mathrm{C+F}}$ rotations)

$$\Gamma \simeq \left(\begin{array}{cc|cc|c} & & \Lambda & & \\ & & & \mathbf{0}_{M-r-2\gamma} & \\ \hline \mathbf{0}_{2\gamma} & & & & \\ & \mathbf{0}_{M-r-2\gamma} & & & \\ \hline & & & & 0 \end{array}\right), \tag{5.118}$$

$$\Lambda \equiv i\sigma_2 \otimes \mathrm{diag}\left(\lambda_1 \mathbf{1}_{p_1}, \cdots, \lambda_q \mathbf{1}_{p_q}\right) , \tag{5.119}$$

with $\lambda_i > \lambda_{i+1} > 0$ and 2γ ($< 2(M-r)+1$) being the rank of Γ ($\gamma = \sum_{i=1}^{q} p_i$). By making use of the V-transformation and the $SO(2M+1)_{\mathrm{C+F}}$ symmetry, we finally obtain the following moduli matrix

$$H_0 = \left(\begin{array}{cc|cc||cc|cc||c} z^2\mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2\mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & z^2\mathbf{1}_{2\gamma} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 \\ 0 & \Lambda' z & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 & 0 \\ \hline 0 & 0 & \Lambda^{-1}z & 0 & 0 & 0 & \mathbf{1}_{2\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 \\ \hline\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{array}\right), \tag{5.120}$$

where we have diagonalized $a_{1;A}$ as

$$a_{1;A} = u\Lambda' u^{\mathrm{T}} , \quad \Lambda' \equiv i\sigma_2 \otimes \mathrm{diag}\left(\lambda'_1 \mathbf{1}_{p'_1}, \cdots, \lambda'_{q'} \mathbf{1}_{p'_{q'}}\right) , \quad u \in U(2\alpha) , \tag{5.121}$$

with 2α being the rank of $a_{1;A}$ and $2\alpha = 2\sum_{i=1}^{q'} p'_i$. Let us now rearrange the eigenvalues $\{\lambda_i^{-1}, \lambda'_i\}$ as

$$\mathrm{diag}(\Lambda', \Lambda^{-1}) \to i\sigma_2 \otimes \mathrm{diag}\left(\tilde{\lambda}_1 \mathbf{1}_{\tilde{p}_1}, \cdots, \tilde{\lambda}_s \mathbf{1}_{\tilde{p}_s}\right) , \quad \tilde{\lambda}_a > \tilde{\lambda}_{a+1} > 0 , \tag{5.122}$$

and redefine $t \equiv r - 2\alpha,\ u \equiv M - r - 2\gamma$ with the constraint:

$$s, t, u \in \mathbb{Z}_{\geq 0} , \quad \tilde{p}_i \in \mathbb{Z}_{>0} , \quad t + u + 2\sum_{i=1}^{s} \tilde{p}_i = M , \tag{5.123}$$

such that the r-dependence in the form of Eq. (5.120) disappears. We conclude that the moduli space of vortices is (apart from the center of mass position):

$$\mathcal{M}_{SO(2M+1)}^{k=1,\mathrm{ori}} = \bigcup_{\{t,u,\tilde{p}_i |\, \mathrm{Eq.}\,(5.123)\}} \mathbb{R}_{>0}^{s} \times \mathcal{O}_{t,u,\tilde{p}_i} , \tag{5.124}$$

$$\mathcal{O}_{t,u,\tilde{p}_i} = \frac{SO(2M+1)}{U(t) \times SO(2u+1) \times \prod_{a=1}^{s} USp(2\tilde{p}_a)} . \tag{5.125}$$

Note that there does not appear any $\mathbb{Z}_2$ factor contrary to the $SO(2M)$ case since

$$P = \mathrm{diag}(1, \cdots, 1, -1) \in O(2M+1)/SO(2M+1) ,$$

acts trivially on H_0 in Eq. (5.120). The special orbits in Eq. (5.111) are obtained simply by choosing $s = 0$. A sequence of the moduli space is given in Fig. 5.9.

At the most generic points, the moduli spaces are locally of the form

$$\mathcal{M}^{k=1,\mathrm{ori}}_{SO(4m+1),+} = \mathbb{R}^m_{>0} \times \frac{SO(4m+1)}{USp(2)^m} , \tag{5.126}$$

$$\mathcal{M}^{k=1,\mathrm{ori}}_{SO(4m+1),-} = \mathbb{R}^{m-1}_{>0} \times \frac{SO(4m+1)}{U(1) \times USp(2)^{m-1} \times SO(3)} , \tag{5.127}$$

$$\mathcal{M}^{k=1,\mathrm{ori}}_{SO(4m+3),+} = \mathbb{R}^m_{>0} \times \frac{SO(4m+3)}{U(1) \times USp(2)^m} , \tag{5.128}$$

$$\mathcal{M}^{k=1,\mathrm{ori}}_{SO(4m+3),-} = \mathbb{R}^m_{>0} \times \frac{SO(4m+3)}{USp(2)^m \times SO(3)} . \tag{5.129}$$

The dimensions of the moduli spaces are then summarized as

$$\dim_{\mathbb{C}} \left[\mathcal{M}^{k=1,\mathrm{ori}}_{SO(2M+1),+}\right] = M^2 , \quad \dim_{\mathbb{C}} \left[\mathcal{M}^{k=1,\mathrm{ori}}_{SO(2M+1),-}\right] = M^2 - 1 . \tag{5.130}$$

5.5.1 Examples: $G' = SO(3), SO(5)$

$k = 1$ **local vortex for** $G' = SO(3)$

Let us discuss the simplest example, *viz.* $G' = SO(3)$. In this model there are two patches having $Q_{\mathbb{Z}_2} = +1$. The moduli matrices take the respective forms

$$H_0^{(1)} = f_3(h^{(1,0)}(0,a)) = \begin{pmatrix} z^2 & 0 & 0 \\ -a^2 & 1 & \sqrt{2}a \\ -\sqrt{2}az & 0 & z \end{pmatrix} , \quad H_0^{(-1)} = f_3(h^{(0,1)}(0,b)) . \tag{5.131}$$

where $h^{(*,*)}(z_0, a)$ are the two patches (5.96) of $\mathcal{M}^{k=1}_{SU(2)}$ and the map f_3 is defined by

$$f_3 : A = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in GL(2,\mathbb{C}) \to f_3(A) = \begin{pmatrix} c^2 & -d^2 & \sqrt{2}cd \\ -e^2 & f^2 & -\sqrt{2}ef \\ \sqrt{2}ce & -\sqrt{2}df & cf + de \end{pmatrix} , \tag{5.132}$$

and expresses the isomorphism $GL(2,\mathbb{C})/\mathbb{Z}_2 \simeq [U(1) \times SO(3)]^{\mathbb{C}}$. On the other hand, there exists just a single patch with $Q_{\mathbb{Z}_2} = -1$. This "patch" actually contains only a point

$$H_0^{(0)} = f_3(\sqrt{z}\mathbf{1}_2) = z\mathbf{1}_3 . \tag{5.133}$$

This vortex does not break the color-flavor symmetry $G'_{\mathrm{C+F}} = SO(3)$: it is an Abelian vortex i.e. not having any orientational moduli. Hence, the moduli spaces $\mathcal{M}^{k=1}_{SO(3),\pm}$ are

$$\mathcal{M}^{k=1}_{SO(3),+} \simeq \mathcal{M}^{k=1}_{SU(2)} \simeq \mathbb{C} \times \mathbb{C}P^1 , \quad \mathcal{M}^{k=1}_{SO(3),-} \simeq \mathbb{C} . \tag{5.134}$$

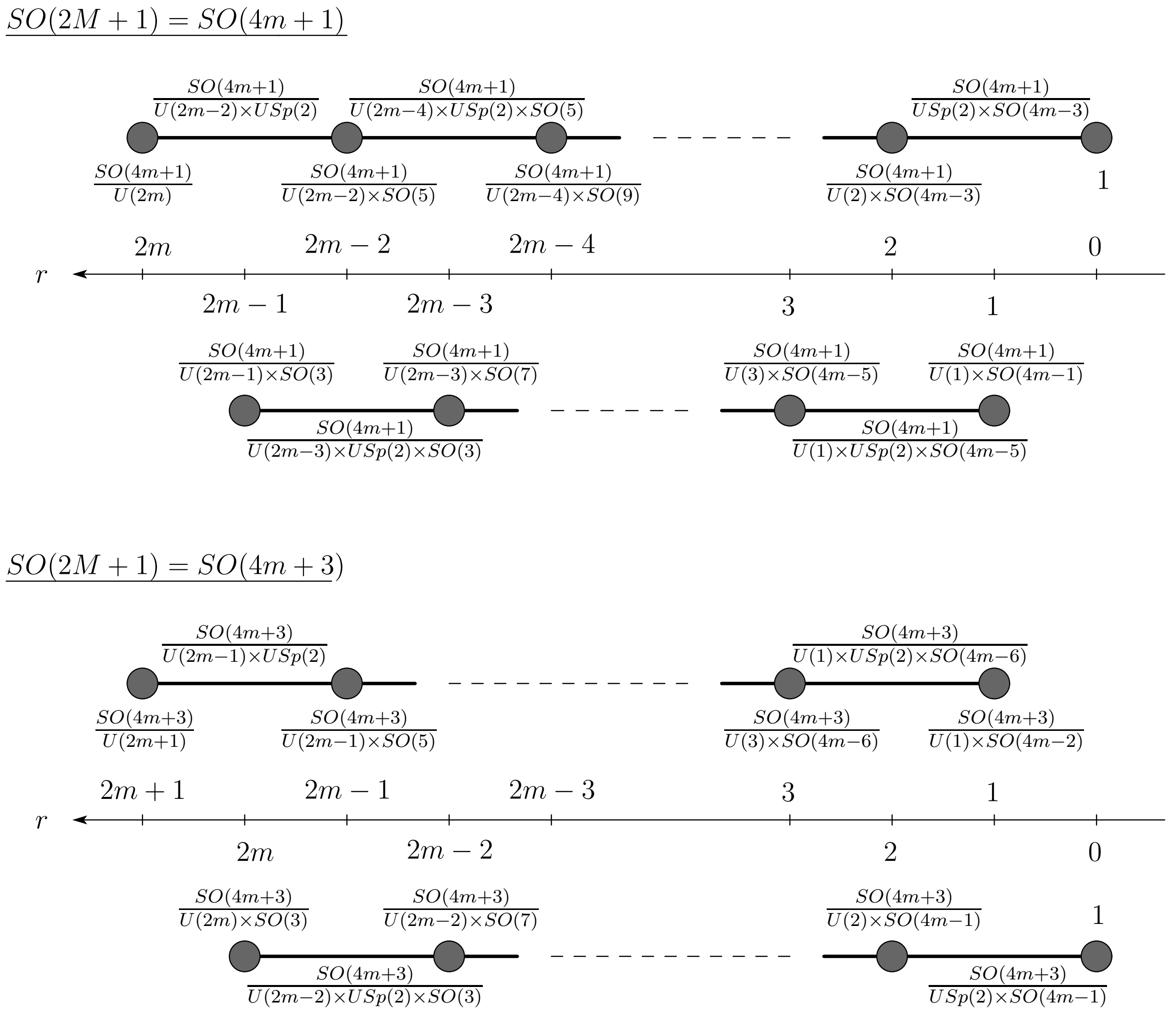


Figure 5.9: Sequences of the $k = 1$ vortices for $SO(4m+1)$ and for $SO(4m+3)$ theories.

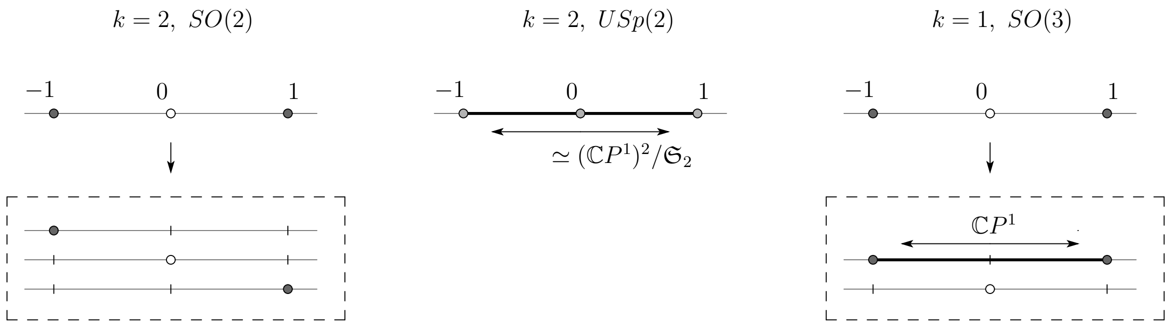


Figure 5.10: $k = 1$, $SO(3)$ and $k = 2$; $SO(2), USp(2)$.

Note that f_3 always maps the moduli matrix of $G' = SU(2)$ to that of $G' = SO(3)$ with $Q_{\mathbb{Z}_2} = +1$.

We have seen very similar dual weight diagrams for $k = 2$, $SO(2), USp(2)$ and $k = 1$, $SO(3)$ vortices. All of them consist of three sites on a straight line. However, when the connectedness is taken into account, they are quite different, see Fig. 5.10.

The three points are isolated in the $SO(2)$ case while they are all connected in the $USp(2)$ case. In the case of $SO(3)$, they split into two diagrams. One is a singlet and the other has two sites mutually connected, which describes $\mathbb{C}P^1$.

$k = 1$ **local vortex for** $G' = SO(5)$

Finally, we move on to the second simplest case of odd SO vortices: $G' = SO(5)$. Let us first list all the patches, starting with those having $Q_{\mathbb{Z}_2} = +1$:

$$H_0^{(0,0)} = z\mathbf{1}_5 + A\ , \tag{5.135}$$

$$H_0^{(1,1)} = \begin{pmatrix} z^2 & 0 & 0 & 0 & 0 \\ 0 & z^2 & 0 & 0 & 0 \\ -c_3^2 & -c_1 z + c_2 - c_3 c_4 & 1 & 0 & \sqrt{2}c_3 \\ c_1 z - c_2 - c_3 c_4 & -c_4^2 & 0 & 1 & \sqrt{2}c_4 \\ -\sqrt{2}c_3 z & -\sqrt{2}c_4 z & 0 & 0 & z \end{pmatrix}, \tag{5.136}$$

where

$$A = \begin{pmatrix} -a_1a_2 - a_3a_4 & -a_4^2 & 0 & a_1^2 & \sqrt{2}a_1a_4 \\ a_3^2 & -a_1a_2 + a_3a_4 & -a_1^2 & 0 & -\sqrt{2}a_1a_3 \\ 0 & a_2^2 & a_1a_2 + a_3a_4 & -a_3^2 & \sqrt{2}a_2a_3 \\ -a_2^2 & 0 & a_4^2 & a_1a_2 - a_3a_4 & \sqrt{2}a_2a_4 \\ -\sqrt{2}a_2a_3 & -\sqrt{2}a_2a_4 & -\sqrt{2}a_1a_4 & \sqrt{2}a_1a_3 & 0 \end{pmatrix}. \tag{5.137}$$

The patches $H_0^{(1,-1)}$, $H_0^{(-1,1)}$ and $H_0^{(-1,-1)}$ can be obtained from $H_0^{(1,1)}$ by the permutations (5.11). This means that the four patches $\{H_0^{(1,1)}, H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}\}$ are on an $SO(5)$ orbit and they are certainly connected. By the general discussion in the previous Section, we know that also $H^{(0,0)}$ and all the other four patches are connected. This can be seen explicitly by studying the transition functions among all these patches:

$$H_0^{(1,1)} = V^{(1,1),(0,0)} H_0^{(0,0)}, \tag{5.138}$$

$$V^{(1,1),(0,0)} = \begin{pmatrix} z + \frac{c_2 + c_3c_4}{c_1} & \frac{c_4^2}{c_1} & 0 & -\frac{1}{c_1} & -\frac{\sqrt{2}c_4}{c_1} \\ -\frac{c_3^2}{c_1} & z + \frac{c_2 - c_3c_4}{c_1} & \frac{1}{c_1} & 0 & \frac{\sqrt{2}c_3}{c_1} \\ 0 & -c_1 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 \\ -\sqrt{2}c_3 & -\sqrt{2}c_4 & 0 & 0 & 1 \end{pmatrix}, \quad \left\{ \begin{array}{l} a_1 = \pm\frac{1}{\sqrt{c_1}} \\ a_i = \pm\frac{c_i}{\sqrt{c_1}}\ , \end{array} \right. \tag{5.139}$$

where $i = 2, 3, 4$ and the same sign has to be chosen for all the transition functions. This means that the moduli space for the minimal vortex with $Q_{\mathbb{Z}_2} = +1$ in $G' = SO(5)$ is

$$\mathcal{M}_{SO(5),+}^{k=1} = \mathbb{C} \times W\mathbb{C}P^4_{(2,1,1,1,1)} \simeq \mathbb{C} \times \mathbb{C}P^4/\mathbb{Z}_2\ , \tag{5.140}$$

where the subscript $(2,1,1,1,1)$ denotes the $U(1)^{\mathbb{C}}$ charges. The weighted complex projective space $W\mathbb{C}P^4_{(2,1,1,1,1)}$ is defined by the following equivalence relation among five complex parameters ϕ_i (i.e. the homogeneous coordinates)

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \sim (\lambda^2\phi_1, \lambda\phi_2, \lambda\phi_3, \lambda\phi_4, \lambda\phi_5)\,, \quad \lambda \in \mathbb{C}^*\,. \tag{5.141}$$

On the other hand, the patches corresponding to $Q_{\mathbb{Z}_2} = -1$ take the form

$$H_0^{(1,0)} = \begin{pmatrix} z^2 & 0 & 0 & 0 & 0 \\ -b_1 z & z & 0 & 0 & 0 \\ -b_1 b_2 - b_3^2 & b_2 & 1 & b_1 & \sqrt{2}b_3 \\ -b_2 z & 0 & 0 & z & 0 \\ -\sqrt{2}b_3 z & 0 & 0 & 0 & z \end{pmatrix}. \tag{5.142}$$

The remaining patches $H_0^{(-1,0)}$, $H_0^{(0,1)}$ and $H_0^{(0,-1)}$ are obtained by permutations (5.11) from $H_0^{(1,0)}$. Since all of them are on an $SO(5)$ orbit, the moduli space of the $k=1$ vortices with $Q_{\mathbb{Z}_2} = -1$ is

$$\mathcal{M}^{k=1}_{SO(5),-} = \mathbb{C} \times \frac{SO(5)}{U(1)\times SO(3)} \simeq \mathcal{M}^{k=1}_{USp(4)} = \mathbb{C} \times \frac{USp(4)}{U(2)}\,. \tag{5.143}$$

The following V-transformation from the $(1,0)$-patch to the $(-1,0)$-patch is

$$H^{(-1,0)}(z) = V^{(-1,0),(1,0)}(z) H^{(1,0)}(z)\,, \tag{5.144}$$

$$V^{(-1,0),(1,0)} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\Xi & 0 & 0 \\ 0 & \frac{c'^2}{\Xi} & a'z & -\frac{2a'^2}{\Xi} & -\frac{2a'c'}{\Xi} \\ -\frac{2}{\Xi} & -\frac{2b'z}{\Xi} & z^2 & -\frac{2a'z}{\Xi} & -\frac{2c'z}{\Xi} \\ 0 & -\frac{2b'^2}{\Xi} & b'z & \frac{c'^2}{\Xi} & -\frac{2b'c'}{\Xi} \\ 0 & -\frac{2b'c'}{\Xi} & c'z & -\frac{2a'c'}{\Xi} & 1-\frac{2c'^2}{\Xi} \end{pmatrix}, \tag{5.145}$$

$$\Xi \equiv 2a'b' + c'^2\,. \tag{5.146}$$

The transition functions are as follows

$$a = -\frac{2a'}{\Xi}\,, \quad b = -\frac{2b'}{\Xi}\,, \quad c = \frac{2c'}{\Xi}\,. \tag{5.147}$$

CHAPTER 6

Semi-local vortices

The structure of the moduli spaces turns out to be considerably richer in the theories with $G' = SO(N)$ and $G' = USp(2M)$ than those of the $U(N)$ theories. We find that vortices are generally of the semi-local type, with power-like tails in their profile functions even in the case with minimal matter content sufficient for having the full Higgs phase with the special color-flavor locked vacuum. The classical moduli spaces will be divided into topological sectors by the $\mathbb{Z}_2$ charge, which corresponds to non-trivial $\pi_1(G')$. These sectors are simply disconnected topologically. Otherwise, all other parts of the moduli space are connected by V-transformations and transition functions which will be studied in detail for $G' = SO(2M)$ and for $G' = USp(2M)$ with $k = 1, 2$ and $G' = SO(2M + 1)$ with $k = 1$.

6.1 Properties of semi-local vortices

We now turn to the more general type of solutions, by relaxing the strong condition (6.15) and obtain the so-called weak condition, which has already been stated in Chap. 1. For $G' = SU(N)$ the condition in terms of the moduli matrix is

$$\det H_0(z) = z^k + \mathcal{O}\left(z^{k-1}\right) \ , \tag{6.1}$$

while in the case of $G' = SO(2M)$ or $G' = USp(2M)$ it is

$$H_0^{\mathrm{T}}(z) J H_0(z) = z^k + \mathcal{O}\left(z^{k-1}\right) \ , \tag{6.2}$$

and for $G' = SO(2M + 1)$ it is instead

$$H_0^{\mathrm{T}}(z) J H_0(z) = z^{2k} + \mathcal{O}\left(z^{2k-1}\right) \ . \tag{6.3}$$

As we will see, this leads to the so-called semi-local vortices. This condition can easily be written down for any other simple group by means of its invariant tensors.

The moduli matrix H_0 has the general properties:

- it is an $N \times N_{\mathrm{F}}$ complex matrix;
- all of its elements are polynomials in z. The algorithm given in Ref. [134] implies that it is sufficient to consider only polynomials as holomorphic functions;

- it is defined only up to the V-equivalence relation, Eq. (1.54);
- it is subject to the weak condition, Eq. (1.72).

The moduli parameters ϕ^i for a BPS vortex solution emerge as coefficients in $H_0(z)$ and thus the moduli space of the solutions is defined by the above properties only. Of course, all the matrices which we found in Sec. 5 for the local vortices satisfy these conditions *a fortiori*. Specifically, one can easily check that the special point $H_0^{(\tilde{\mu}_1,\cdots,\tilde{\mu}_M)}$ in Eq. (4.8) satisfies the weak condition.

In the strong coupling limit $e, g \to \infty$, the master equations (1.58) and (1.59) are exactly solved by $\Omega' = \Omega'_\infty, \omega = \omega_\infty$ in Eq. (1.93) and the energy density and Kähler potential for the effective action for the vortices (lumps) are given by [5]

$$\mathcal{E} = 2\partial\bar{\partial}\mathcal{K}\,, \quad K(\phi^i, \phi^{i*}) = \int d^2x\, \mathcal{K}\,, \quad \mathcal{K} = \xi \log \mathrm{Tr}\left[\sqrt{I_{G'} I_{G'}^\dagger}\right]\,, \tag{6.4}$$

with the G'-invariant $I_{G'} = H_0^{\mathrm{T}}(z) J H_0(z)$. Even in the case of finite gauge couplings, these are considered to be good approximations when $m_{e,g} L \gg 1$ where L is the typical distance from the core of the vortices. By substituting a typical form of $H_0(z)$ into the above formula, one can obtain multiple peaks in the energy profile even for a minimal winding vortex $(k = 1)$. We call these interesting multi-peak solutions fractional vortices and they are discussed in Chap. 10. First we will discuss the differences between local and semi-local vortices in the next Section and then we will move on to solving some technical problems before studying the patches and transition functions for the semi-local vortices.

6.2 Local versus semi-local vortices

One is often interested in knowing which of the moduli parameters describe the so-called local (or ANO-) vortices [11, 12]), which have profile functions with exponential tails. For example, the thoroughly studied $U(N)$ non-Abelian vortices are of the local type when the model has a unique vacuum: this is indeed the case when the number of flavors is the minimal one, i.e. just sufficient for the color-flavor locked vacuum ($N_{\mathrm{F}} = N$ Higgs fields in the $\mathbf{N}$ representation of $SU(N)$). For N_{F} greater than N, the vacuum moduli space contains continuous moduli

$$Gr_{N_{\mathrm{F}},N} \simeq SU(N_{\mathrm{F}})/[SU(N_{\mathrm{F}} - N) \times SU(N) \times U(1)]\,, \tag{6.5}$$

and, as a consequence, the generic non-Abelian vortex solution is of the "semi-local" type [84, 85, 86, 48, 227], with power-like tails. A characteristic feature of the semi-local vortices is their size moduli, which are non-normalizable [115, 116]. A lesson from the $U(N)$ non-Abelian vortices is that the semi-local vortices become local (ANO-like) vortices, when all the size moduli are set to zero.

Our model with $G' = SO(N)$ or $USp(2M)$, even with our choice $N_{\mathrm{F}} = N$, that is the minimum number of flavors that allows for a color-flavor locked vacuum, possesses always a non-trivial vacuum moduli space. In fact, in the class of theories considered here, its dimension is given by the following general formula

$$\dim_{\mathbb{C}}\left[\mathcal{M}_{\rm vac}\right] = NN_{\rm F} - \dim_{\mathbb{C}}\left[U(1)^{\mathbb{C}} \times G'^{\mathbb{C}}\right] > 0\,. \tag{6.6}$$

This strongly suggests that even for $N_{\rm F} = N$, generic configurations are of the "semi-local" type. The Kähler metric and its potential on the vacuum moduli space have been obtained in Ref. [5].

The distinction between local and semi-local vortices can be made by using the moduli matrix. In order to see this, the asymptotic behavior of the configurations must be clarified. First note that the vacuum moduli spaces of our models are Kähler manifolds $\mathcal{M}_{\rm vac}$ and our gauge theories reduce to non-linear σ models (NLσMs) whose target space is $\mathcal{M}_{\rm vac}$, when the gauge couplings are sent to infinity. In this limit, vortices generally reduce to the so-called σ model lumps [71] (sometimes also called two-dimensional Skyrmions or σ model instantons) characterized by

$$\pi_2(\mathcal{M}_{\rm vac})\,,$$

i.e. a wrapping around a 2-cycle inside $\mathcal{M}_{\rm vac}$. By rescaling sizes, taking the strong coupling limit can be interpreted as picking up the asymptotic behavior, and thus, even for a finite gauge coupling, asymptotic configurations of semi-local vortices are well-approximated by lumps [86, 48, 227].

Consider the lump solutions of the NLσM on $\mathcal{M}_{\rm vac}$. Let us take holomorphic G-invariants $\{I_G^I\}$ as inhomogeneous coordinates of $\mathcal{M}_{\rm vac}$ and denote its Kähler potential by $K = K(I_G, I_G^*)$. A lump solution is then given by a holomorphic map

$$z \in \mathbb{C} \quad \rightarrow \quad I_G^I = f^I(z) \in \mathcal{M}_{\rm vac}\,. \tag{6.7}$$

with single-valued functions $\{f^I(z)\}$. For finite-energy solutions, the boundary $|z| = \infty$ is mapped to a single point $I_G^I = v^I \in \mathcal{M}_{\rm vac}$. So the maps $\{f^I(z)\}$ are asymptotically of the form

$$f^I(z) = v^I + \frac{u^I}{z} + \mathcal{O}\left(z^{-2}\right), \quad u^I \in \mathbb{C}\,. \tag{6.8}$$

The corresponding energy density $\mathcal{E}$ has a power behavior

$$\mathcal{E} = 2K_{J\bar{J}}(I_G, \bar{I}_G)\,\partial I_G^J(z)\,\bar{\partial}\bar{I}_G^{\bar{J}}(\bar{z}) = \frac{2}{|z|^4}\,K_{J\bar{J}}(v,\bar{v})\,u^J\bar{u}^{\bar{J}} + \mathcal{O}\left(|z|^{-5}\right)\,, \tag{6.9}$$

where we assume that $\{I_G^I\}$ is a local coordinate system in the vicinity of the point $I_G^I = v^I$ and the manifold is smooth at that point. As mentioned above, this asymptotic behavior is valid for that of the vortices as well. Since $\{I_G^I\} \simeq \{I_{G'}^i\}/U(1)^{\mathbb{C}}$ in the case $G = G' \times U(1)$, the holomorphic maps and the moduli matrix are related by

$$\{f^I(z)\} \simeq \{I_{G'}(z)\}/\sim\,, \tag{6.10}$$

where "$\sim$" is defined as the equivalence relation

$$I_{G'}^i(z) \sim P(z) I_{G'}^i(z)\,, \qquad \text{with } P(z) \in \mathbb{C}[z]\,. \tag{6.11}$$

Hence, the asymptotic tail of the configurations is generically power-like, i.e. the generic vortices are of the semi-local type.

Although this is in general the case, it might happen that all the holomorphic functions $\{I^i_{G'}(H_0(z))\}$ have common zeros and that the quotient above is ill-defined. In such case, from the point of view of $f^I(z)$, we completely lose the information about the common zeros accompanied by some vorticity. Namely, the signature of the corresponding vortices vanishes from their polynomial tails and $\pi_2(\mathcal{M}_{\rm vac})$ becomes trivial[1]. Specifically, it can happen that all the holomorphic invariants are proportional to a polynomial $P(z)$:

$$f^I(z) = \text{const.} \quad \Leftarrow \quad I^i_{G'}(H_0(z)) = P(z)^{\frac{n_i}{n_0}} \quad \text{for all } i\,, \tag{6.12}$$

or possibly that there exists only one such holomorphic invariant. In the case of the $U(N)$, with $N_{\rm F} = N$ i.e. the model considered earlier, $\mathcal{M}_{\rm vac}$ is just a single point. Even in the SO and USp cases, we do not consider any non-trivial element of the second homotopy group of $\mathcal{M}_{\rm vac}$ but we fix a point of $\mathcal{M}_{\rm vac}$ at $|z| \to \infty$. Therefore, one must return to the master equations to examine the asymptotic behavior. The moduli matrix satisfying Eq. (6.12) could be transformed to a trivial one such that $\Omega_0 = \mathbf{1}_N$ in Eqs. (1.58) and (1.59), by using an extended V-transformation allowing for negative powers of z, with a singular determinant $\det\left(V(z)\right) = P(z)^{-1}$. After this operation the master equation would take the form of a Liouville-type equation with point-like sources;[2] hence the asymptotic tail is indeed exponential. In other words, the conditions (6.12) mean that the (static) vortex is decoupled from any massless mode in the Higgs vacuum and hence the dominant contribution to its configuration comes from massive modes in the bulk. The corresponding vortices are purely of local type. Conversely, we can clearly identify a local vortex and its position by looking at common zeros, although a composite state of a semi-local vortex and a local vortex also has a polynomial tail. The above observations can briefly be summarized as follows. The asymptotic behavior of a vortex is classified by the lightest modes in the bulk coupled to its configuration. In other words, a vortex is necessarily of the local type, when the vacuum moduli space is just a point (i.e. a unique vacuum). Semi-local vortices are present only if the vacuum moduli space is non-trivial (i.e. having continuous moduli).

Once we have clarified the origin of the of polynomial tails, it is easier to identify the non-normalizable modes and the results in Ref. [5] for lumps can be readily applied to vortices. Semi-local vortices always have non-normalizable moduli, which live on the tangent bundle of the moduli space of vacua[3]

$$(v^I, u^I) \in T\mathcal{M}_{\rm vac}\,. \tag{6.13}$$

[1]The price of the loss of vorticity in the map (6.7) is the appearance of small lump singularities, which manifest themselves as spikes (delta functions) in the energy density.

[2] In the well-known Abelian case $G = U(1)$, this transformed master equation is nothing but Taubes' equation. This transformation for non-Abelian cases means that all information about orientational moduli are also localized at the zeros, in other words, the moduli matrix can be reconstructed from the data at the zeros in the case of local vortices [134]. For semi-local vortices, this is clearly not the case.

[3]v^I are nothing but vacuum moduli and all of the u^I's are not always independent and consist of overall semi-local moduli like an overall size modulus. The interpretation as a tangent bundle can be derived from Eq. (6.8)

In our case, $G' = SO(N), USp(N)$, with the common $U(1)$ charge for the scalar fields H, all the $G^{\mathbb{C}}$ invariants $I_G^I(H)$ can be written using the meson $I_{SO,USp}$ in Eq. (1.94). For instance, since $\mathrm{Tr}[I_{SO,USp}] \neq 0$ in the chosen vacuum, we can construct

$$I_G^{(r,s)}(H) \equiv \frac{(I_{SO,USp}(H))^r{}_s}{\mathrm{Tr}[I_{SO,USp}(H)]} = \frac{\left(H^{\mathrm{T}}JH\right)^r{}_s}{\mathrm{Tr}[H^{\mathrm{T}}JH]} , \qquad 1 \leq r \leq s \leq N . \tag{6.14}$$

The condition for (winding k) local vortices is thus:

$$I_{SO,USp}(H_0) = H_0^{\mathrm{T}}(z)JH_0(z) = \left(\prod_{i=1}^{k}(z - z_i)^{\frac{2}{n_0}}\right) J . \tag{6.15}$$

This is called the strong condition, in contrast to the weak condition (1.72) which characterizes a more general class of solutions including semi-local vortices.

One can regard this condition as being physically required by modifying our model in such a way that the continuous directions of the vacuum are indeed being lifted. For instance, it is not difficult to add an appropriate superpotential δW to our model, introducing a chiral multiplet A which is a traceless N-by-N matrix taking value in the $\mathfrak{usp}$ ($\mathfrak{so}$) algebra in the SO case (USp case), viz. $A^{\mathrm{T}}J = JA$, and having a $U(1)$ charge -2:

$$\delta W \propto \mathrm{Tr}[AH^{\mathrm{T}}JHJ] , \tag{6.16}$$

however such a term would nevertheless reduce the amount of supersymmetry. As we have seen in Chap. 5 in some cases, the strong condition can give rise to singularities in the moduli space, which will be inherited into the target space of an effective action for the local vortices.

6.3 Dimension of the moduli space

The index theorem which is demonstrated in Chap. 7 tells us that our moduli space has dimension:

$$\dim_{\mathbb{C}} (\mathcal{M}_{G',k}) = \frac{kN^2}{n_0} = \nu N^2 . \tag{6.17}$$

This dimension should coincide with that of the space spanned by the moduli in $H_0(z)$, if the master equations have a unique solution for a given $H_0(z)$. It is easy to confirm this by considering the vicinity of a special point of the moduli space.

Let us find the general form of H_0 in the vicinity of the special point (4.8) by perturbing H_0. For definiteness, let us consider the perturbation around $H_0^{(\frac{k}{2},\cdots,\frac{k}{2})}$:

$$H_0^{(\frac{k}{2},\cdots,\frac{k}{2})} + \delta H_0 = \begin{pmatrix} z^k\mathbf{1}_M & \\ & \mathbf{1}_M \end{pmatrix} + \begin{pmatrix} \delta A(z) & \delta C(z) \\ \delta B(z) & \delta D(z) \end{pmatrix} , \tag{6.18}$$

where $\delta A(z), \delta B(z), \delta C(z)$ and $\delta D(z)$ are $M \times M$ matrices whose elements are holomorphic functions of z with small (infinitesimal) coefficients[4]. Not all of the fluctuations are

[4]Notice that here we are considering fluctuations around a k-vortex configuration with even parity. The generalization to the odd case is discussed at the end of the Section.

independent though: we must fix them uniquely by using the V-equivalence (1.54). The infinitesimal V-transformation satisfies the condition (where we have set $V = \mathbf{1}_N$)

$$\delta V^{\mathrm{T}}(z)J + J\delta V(z) = 0 ,$$

which just represents the algebra of $SO(2M, \mathbb{C}), USp(2M, \mathbb{C})$ and can be expressed as

$$\delta V(z) = \begin{pmatrix} \delta L(z) & \delta N_{A,S}(z) \\ \delta M_{A,S}(z) & -\delta L^{\mathrm{T}}(z) \end{pmatrix} . \tag{6.19}$$

Again $\delta L(z), \delta M_{A,S}(z)$ and $\delta N_{A,S}(z)$ are $M \times M$ matrices whose elements are holomorphic in z and their coefficients are infinitesimally small. Acting with this infinitesimal V-transformation on the moduli matrix

$$\delta V(z) H_0^{(\frac{k}{2},\cdots,\frac{k}{2})} + \delta H_0 \simeq \begin{pmatrix} z^k \delta L(z) & \delta N_{A,S}(z) \\ z^k \delta M_{A,S}(z) & -\delta L^{\mathrm{T}}(z) \end{pmatrix} + \begin{pmatrix} \delta A(z) & \delta C(z) \\ \delta B(z) & \delta D(z) \end{pmatrix} , \tag{6.20}$$

we can set $\delta D(z) \to 0$, $\delta C \to \delta C_{S,A}(z)$ and $\delta B(z) \to \delta B_{S,A}(z) + \delta b(z)$ yielding:

$$\delta H_0 = \begin{pmatrix} \delta A(z) & \delta C_{S,A}(z) \\ \delta B_{S,A}(z) + \delta b(z) & 0 \end{pmatrix} . \tag{6.21}$$

Note that we have adopted the notation that $\delta X(z)$ stands for a general polynomial function while $\delta x(z)$ denotes a holomorphic function whose degree is less than the vortex number k. Now the V-transformation is completely fixed, and one can determine the true degrees of freedom of the fluctuations. The infinitesimal form of the weak condition (1.72) is

$$\delta H_0^{\mathrm{T}}(z) J H_0(z) + H_0(z) J \delta H_0(z) = \mathcal{O}(z^{k-1}) .$$

This leads to $\delta A \to \delta a(z)$, $\delta C_{S,A}(z) \to \delta c_{S,A}(z)$, $\delta B_{S,A}(z) \to 0$ and $\delta b(z) \to \delta b_{A,S}(z)$:

$$\delta H_0(z) = \begin{pmatrix} \delta a(z) & \delta c_{S,A}(z) \\ \delta b_{A,S}(z) & 0 \end{pmatrix} . \tag{6.22}$$

These are good coordinates in the vicinity of the special point

$$H_0^{(\frac{k}{2},\cdots,\frac{k}{2})} = \mathrm{diag}\left(z^k, \cdots, z^k, 1, \cdots, 1\right) .$$

Of course, this is a only local description but it is sufficient for counting the dimensions of the moduli space. The complex dimension is the number of the complex parameters in the fluctuations

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-semi-local}}_{SO(2M),USp(2M)} = 2kM^2 . \tag{6.23}$$

In order to restrict the solutions to the local vortices, one further imposes the following conditions:

$$\delta a(z) \to \delta P(z) \mathbf{1}_M , \quad \delta c_{S,A}(z) \to 0 , \tag{6.24}$$

with an arbitrary polynomial $\delta P(z)$ of order $(k-1)$. This leads to the dimension of the k local vortex moduli:

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-local}}_{SO(2M),+} = k\left(1 + \frac{M(M-1)}{2}\right) , \tag{6.25}$$

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-local}}_{USp(2M)} = k\left(1 + \frac{M(M+1)}{2}\right) . \tag{6.26}$$

In a similar way, one can count the dimension in the vicinity of the special point of positive chirality $(k, \cdots, k)$[5] for the $SO(2M+1)$ case and obtain

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-semi-local}}_{SO(2M+1),+} = k\,(2M+1)^2 , \tag{6.27}$$

$$\dim_{\mathbb{C}} \mathcal{M}^{k\text{-local}}_{SO(2M+1),+} = k\left(M^2+1\right) . \tag{6.28}$$

Notice that these results can be considered as a non-trivial consistency check for the moduli matrix formalism. In fact, by physical arguments, we always expect the following relation among the dimensions of the moduli spaces:

$$\dim_{\mathbb{C}} \mathcal{M}_k = k \dim_{\mathbb{C}} \mathcal{M}_{k=1} , \tag{6.29}$$

which is valid both in the local and semi-local case. This relation can readily be used to generalize the above equations to the other cases, including special points with odd chirality.

6.4 The $k = 1$ semi-local vortex for $G' = SO(2M)$ and $G' = USp(2M)$

Let us study the minimal-winding semi-local vortex in this Section. The $k = 1$ vortex is special in the sense that all the fluctuations in Eq. (6.22) can actually be promoted to finite parameters. Namely, the $H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}$-patch is obtained by just replacing the small fluctuations $\delta a(z), \delta b_{A,S}(z), \delta c_{S,A}(z)$ by finite constant parameters $A, B_{A,S}, C_{S,A}$, respectively:

$$H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}(z) = \begin{pmatrix} z\mathbf{1}_M + A & C_{S,A} \\ B_{A,S} & \mathbf{1}_M \end{pmatrix} . \tag{6.30}$$

One can verify that this indeed satisfies the weak condition (1.72) for $k = 1$. Notice that the above matrix can also be rewritten as

$$H_0^{(\frac{1}{2},\cdots,\frac{1}{2})}(z) = \tilde{U}_C \begin{pmatrix} z\mathbf{1}_M + \tilde{A} & \\ & \mathbf{1}_M \end{pmatrix} U_B , \tag{6.31}$$

[5] Around other special points this strategy may not work in the local case. Other special points may sit on an intersection of two different submanifolds and one cannot make a distinction between the fluctuations among them. It is possible, in any case, to identify, case by case, a special point which does not lie on an intersection. However, one might sometimes need to include quadratic fluctuations, in order to implement correctly the strong condition.

where we have defined

$$\tilde{A} \equiv A - C_{S,A}B_{A,S}\ , \quad U_B \equiv \begin{pmatrix} \mathbf{1}_M & \\ B_{A,S} & \mathbf{1}_M \end{pmatrix}\ , \quad \tilde{U}_C \equiv \begin{pmatrix} \mathbf{1}_M & C_{S,A} \\ & \mathbf{1}_M \end{pmatrix}\ . \tag{6.32}$$

When A is proportional to the unit matrix and $C_{S,A}$ is zero, that is, corresponding to a local vortex (5.9), U_B corresponds to the Nambu-Goldstone modes associated with the symmetry breaking $G'_{\rm C+F} \to U(M)$. It is remarkable that this is not always the case for general semi-local configurations since a non-vanishing $\tilde{A}$ and $C_{S,A}$ break $U(M)$ further down. In general, the symmetry breaking is $G'_{\rm C+F} \to \mathbb{Z}_{n_0}$.

Let us next consider the transition functions between two different patches. As we did for the local vortices in Sec. 5, the other patches can be obtained as in Eq. (5.10), i.e. via the permutation matrix P_r defined in Eq. (5.11). Transition functions are always obtained by means of the V-transformations as in Eq. (1.54)

$$H_0'(z) = V(z)H_0(z)\ , \quad V(z) \equiv V_e\, V'(z)\ , \quad V_e \in \mathbb{C}^*\ ,\ V'(z) \in G'^{\mathbb{C}}\ . \tag{6.33}$$

For example, consider two patches, $H_0^{(\frac12,\cdots,\frac12)}(z)$ given by Eq. (6.31) and

$$\begin{aligned} H_0^{(\overbrace{-\frac12,\cdots,-\frac12}^{r},\overbrace{\frac12,\cdots,\frac12}^{M-r})}(z) &= P_r^{-1} H_0^{(\frac12,\cdots,\frac12)\prime}(z) P_r\ , & (6.34)\\ H_0^{(\frac12,\cdots,\frac12)\prime}(z) &= \tilde{U}_{C'} \begin{pmatrix} z\mathbf{1}_M + \tilde{A}' & \\ & \mathbf{1}_M \end{pmatrix} U_{B'}\ . & (6.35) \end{aligned}$$

The equation (6.33) in this case reads

$$\begin{pmatrix} z\mathbf{1}_M + \tilde{A}' & \\ & \mathbf{1}_M \end{pmatrix} U_{B'} P_r U_{-B} = \tilde{U}_{-C'} P_r V \tilde{U}_C \begin{pmatrix} z\mathbf{1}_M + \tilde{A} & \\ & \mathbf{1}_M \end{pmatrix}\ . \tag{6.36}$$

The transition functions will be determined by this condition together with

$$(U_{B'}P_rU_{-B})^{\rm T} J (U_{B'}P_rU_{-B}) = J\ , \quad \text{and} \quad (P_rV)^{\rm T} J (P_rV) = J\ .$$

The solution to these conditions are of the form

$$U_{B'}P_rU_{-B} = \begin{pmatrix} a & a\, d_{A,S} \\ 0 & (a^{-1})^{\rm T} \end{pmatrix}\ , \quad \tilde{U}_{-C'}P_rV\tilde{U}_C = \begin{pmatrix} a & (z\mathbf{1}_M + \tilde{A}')\, a\, d_{A,S} \\ 0 & (a^{-1})^{\rm T} \end{pmatrix}\ , \tag{6.37}$$

with $a \in GL(M,\mathbb{C})$ and $d_{A,S}$ is an $M \times M$ (anti)symmetric matrix and

$$\begin{aligned} \tilde{A}' &= a\,\tilde{A}\,a^{-1}\ , & (6.38)\\ C'_{S,A} &= a\left[C_{S,A} - \frac12\left(\tilde{A}\,d_{A,S} - d_{A,S}\,\tilde{A}^{\rm T}\right)\right]a^{\rm T}\ . & (6.39) \end{aligned}$$

Notice that Tr$\tilde{A}$ is invariant. The final step is to determine a, $d_{A,S}$ and the transition function for $B'_{A,S}$ by investigating the concrete form of U_B

$$U_B = \begin{pmatrix} \mathbf{1}_r & & & \\ & \mathbf{1}_{M-r} & & \\ b_1 & b_2 & \mathbf{1}_r & \\ -\epsilon\, b_2^{\rm T} & b_3 & & \mathbf{1}_{M-r} \end{pmatrix}\ , \quad b_{1,3}^{\rm T} = -\epsilon\, b_{1,3}\ , \tag{6.40}$$

and analogously for $U_{B'}$. Plugging this into the left hand side of the first equation in (6.37), one obtains the following result:

$$a = \begin{pmatrix} -\epsilon\, b_1 & -\epsilon\, b_2 \\ 0 & \mathbf{1}_{M-r} \end{pmatrix} , \quad d_{A,S} = \begin{pmatrix} -b_1^{-1} & \\ & \mathbf{0}_{M-r} \end{pmatrix} . \tag{6.41}$$

The transition functions between $B_{A,S}$ and $B'_{A,S}$ are indeed the same as those of the local vortex in Eq. (5.14)

$$b'_1 = \epsilon\, b_1^{-1} , \quad b'_2 = b_1^{-1} b_2 , \quad b'_3 = b_3 + \epsilon\, b_2^{\mathrm{T}} b_1^{-1} b_2 . \tag{6.42}$$

We again observe an important result from the first equation in (6.37). It tells us that

$$\det P_r = +1 , \tag{6.43}$$

thus there exist two copies of the moduli space, which are disconnected even in the larger space including the semi-local vortices, in the case of $G' = SO(2M)$. It is of course due to the $\mathbb{Z}_2$ parity (see Sec. 4.2). As in the case of the local vortices in $G' = SO(2M)$ theory discussed earlier, the patches with different $\mathbb{Z}_2$-parity are disconnected.

6.4.1 Example: $G' = SO(4)$

Let us give an example in the $G' = SO(4)$ theory. The patches with $\mathbb{Z}_2$-parity $+1$ are

$$H_0^{(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} z+a & b & e & f \\ c & z+d & f & g \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} , \quad H_0^{(-\frac{1}{2},-\frac{1}{2})} = \begin{pmatrix} 1 & 0 & 0 & i' \\ 0 & 1 & -i' & 0 \\ e' & f' & z+a' & b' \\ f' & g' & c' & z+d' \end{pmatrix} . \tag{6.44}$$

These patches are connected by the V-transformation (1.54)

$$H_0^{(-\frac{1}{2},-\frac{1}{2})} = V^{(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2})} H_0^{(\frac{1}{2},\frac{1}{2})} , \tag{6.45}$$

$$V^{(-\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} 0 & 0 & 0 & i' \\ 0 & 0 & -i' & 0 \\ 0 & \frac{1}{i'} & z+\frac{a'+d'}{2} & 0 \\ -\frac{1}{i'} & 0 & 0 & z+\frac{a'+d'}{2} \end{pmatrix} , \tag{6.46}$$

The explicit form of the transition functions (the relation between the primed and unprimed parameters) is given in Eq. (D.1).

There are two more patches for the vortex with $\mathbb{Z}_2$-parity -1 and they are described by the moduli matrices

$$H_0^{(\frac{1}{2},-\frac{1}{2})} = \begin{pmatrix} z+a'' & f'' & e'' & b'' \\ -i'' & 1 & 0 & 0 \\ 0 & 0 & 1 & i'' \\ c'' & g'' & f'' & z+d'' \end{pmatrix} , \quad H_0^{(-\frac{1}{2},\frac{1}{2})} = \begin{pmatrix} 1 & i''' & 0 & 0 \\ f''' & z+d''' & b''' & e''' \\ g''' & c''' & z+a''' & f''' \\ 0 & 0 & -i''' & 1 \end{pmatrix} . \tag{6.47}$$

These two patches are connected in the same way as the two with positive chirality. In fact they define another copy of the same space. In agreement with the general results found above, neither one of the even patches: $H_0^{(-\frac{1}{2},-\frac{1}{2})}$, $H_0^{(\frac{1}{2},\frac{1}{2})}$, is connected with one of the odd, $H_0^{(\frac{1}{2},-\frac{1}{2})}$ and $H_0^{-(\frac{1}{2},\frac{1}{2})}$. One can easily see that there does not exist any V-transformation connecting them. One may construct a holomorphic matrix $X(z)$ which satisfies, for example, $H_0^{(\frac{1}{2},-\frac{1}{2})} = X(z)H_0^{(\frac{1}{2},\frac{1}{2})}$, however, violating the condition $X(z) \in SO(4,\mathbb{C})$.

6.5 The $k = 2$ semi-local vortices

Consider now the patches associated with the $k = 2$ (doubly-wound) vortices. Let us begin with infinitesimal fluctuations around the special point

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})} = \begin{pmatrix} z^2\mathbf{1}_r & & & \\ & z\mathbf{1}_{M-r} & & \\ & & \mathbf{1}_r & \\ & & & z\mathbf{1}_{M-r} \end{pmatrix} \to H_0^{(1,\cdots,1,0,\cdots,0)} + \delta H_0(z)\ . \tag{6.48}$$

In order to get rid of the unphysical degrees of freedom in the fluctuations δH_0, let us consider an infinitesimal V-transformation (1.54)

$$\delta V = \begin{pmatrix} \delta K_{11} & \delta M_{11} & \delta K_{12;A,S} & \delta M_{12} \\ \delta L_{11} & \delta N_{11} & -\epsilon\,\delta M_{12}^{\mathrm{T}} & \delta N_{12;A,S} \\ \delta K_{21;A,S} & \delta M_{21} & -\delta K_{11}^{\mathrm{T}} & -\delta L_{11}^{\mathrm{T}} \\ -\epsilon\,\delta M_{21}^{\mathrm{T}} & \delta N_{21;A,S} & -\delta M_{11}^{\mathrm{T}} & -\delta N_{11}^{\mathrm{T}} \end{pmatrix}\ . \tag{6.49}$$

Acting with the V-transformation on the perturbed moduli matrix, we find

$$\delta H_0 \sim \delta H_0 + \delta V H_0^{(1,\cdots,1,0,\cdots,0)}\ . \tag{6.50}$$

Since the explicit form of $\delta V H_0^{(1,\cdots,1,0,\cdots,0)}$ is

$$\delta V H_0^{(1,\cdots,1,0,\cdots,0)} = \begin{pmatrix} z^2\delta K_{11} & z\delta M_{11} & \delta K_{12;A,S} & z\delta M_{12} \\ z^2\delta L_{11} & z\delta N_{11} & -\epsilon\,\delta M_{12}^{\mathrm{T}} & z\delta N_{12;A,S} \\ z^2\delta K_{21;A,S} & z\delta M_{21} & -\delta K_{11}^{\mathrm{T}} & -z\delta L_{11}^{\mathrm{T}} \\ -z^2\epsilon\,\delta M_{21}^{\mathrm{T}} & z\delta N_{21;A,S} & -\delta M_{11}^{\mathrm{T}} & -z\delta N_{11}^{\mathrm{T}} \end{pmatrix}\ ,$$

the physical degrees of freedom in the fluctuations can be expressed as

$$\delta H_0 = \tag{6.51}$$
$$\begin{pmatrix} \delta A_{11} & \delta C_{11} & \delta A_{12;S,A} & \delta C_{12} \\ \delta B_{11} & \delta D_{11} & 0 & \delta D_{12;S,A} + \delta d_{12;A,S} \\ \delta A_{21;S,A} + \delta a_{21;A,S}^{(1)} z + \delta a_{21;A,S}^{(0)} & \delta c_{21} & 0 & \delta c_{22} \\ \delta B_{21} & \delta D_{21;S,A} + \delta d_{21;A,S} & 0 & \delta d_{22} \end{pmatrix}\ ,$$

where δX denotes a generic holomorphic polynomial and δx stands for a constant matrix. The infinitesimal version of the weak condition (1.72)

$$\delta H_0^{\mathrm{T}}(z) J H_0(z) + H_0(z) J \delta H_0(z) = \mathcal{O}(z)\ , \tag{6.52}$$

turns out to be equivalent to the following conditions

$$\begin{gathered}\{\delta D_{11},\ \delta D_{21;S,A},\ \delta D_{12;S,A}\} = \mathcal{O}(1)\ ,\\ \{\delta A_{11},\ \delta C_{11},\ \delta A_{12;S,A},\ \delta C_{12}\} = \mathcal{O}(z)\ ,\\ \delta A_{21;S,A} = 0\ ,\quad \delta B_{11} = -\delta c_{22}^{\mathrm{T}} z + \delta b_{11}\ ,\quad \delta B_{21} = -\epsilon\,\delta c_{21}^{\mathrm{T}} z + \delta b_{21}\ .\end{gathered} \tag{6.53}$$

We thus find the generic form of the fluctuations in the vicinity of the special point described by the moduli matrix $H_0^{(1,\cdots,1,0\cdots,0)}$ as

$$\delta H_0 = \begin{pmatrix} \delta a_{11}^{(1)} z + \delta a_{11}^{(0)} & \delta c_{11}^{(1)} z + \delta c_{11}^{(0)} & \delta a_{12;S,A}^{(1)} z + \delta a_{12;S,A}^{(0)} & \delta c_{12}^{(1)} z + \delta c_{12}^{(0)} \\ -\delta c_{22}^{\mathrm{T}} z + \delta b_{11} & \delta d_{11} & 0 & \delta d_{12} \\ \delta a_{21;A,S}^{(1)} z + \delta a_{21;A,S}^{(0)} & \delta c_{21} & 0 & \delta c_{22} \\ -\epsilon\,\delta c_{21}^{\mathrm{T}} z + \delta b_{21} & \delta d_{21} & 0 & \delta d_{22} \end{pmatrix} . \tag{6.54}$$

Let us count the dimensions of the moduli space. We have six matrices $\delta a_{ij}^{(\alpha)}$ of size $r \times r$, two matrices δb_{ij} of size $(M-r)\times r$, six matrices $\delta c_{ij}^{(\alpha)}$ of size $r\times(M-r)$ and four matrices δd_{ij} of the size $(M-r)\times(M-r)$. Thus summing up we obtain the correct dimension

$$\dim_{\mathbb{C}}\left[\mathcal{M}^{\text{2-semi-local}}_{SO(2M),USp(2M)}\right] = 4M^2\ . \tag{6.55}$$

The next task is to find the coordinate patches with *finite* parameters (i.e. large fluctuations). To this end, let us naïvely promote all the small fluctuations in Eq. (6.54) to finite parameters as $\delta x \to x$ (as was done in the case of the minimal semi-local vortices) :

$$H_0 = \begin{pmatrix} z^2\mathbf{1}_r + a_{11}^{(1)} z + a_{11}^{(0)} & c_{11}^{(1)} z + c_{11}^{(0)} & a_{12;S,A}^{(1)} z + a_{12;S,A}^{(0)} & c_{12}^{(1)} z + c_{12}^{(0)} \\ -c_{22}^{\mathrm{T}} z + b_{11} & z\mathbf{1}_{M-r} + d_{11} & 0 & d_{12} \\ a_{21;A,S}^{(1)} z + a_{21;A,S}^{(0)} & c_{21} & \mathbf{1}_r & c_{22} \\ -\epsilon\, c_{21}^{\mathrm{T}} z + b_{21} & d_{21} & 0 & z\mathbf{1}_{M-r} + d_{22} \end{pmatrix} . \tag{6.56}$$

But such a procedure is inconsistent with the weak condition (1.72). Although

$$H_0^{\mathrm{T}} J H_0\big|_{\mathcal{O}(z^n)} = 0\ ,$$

for $n \geq 3$, the terms of order $\mathcal{O}(z^2)$ turn out to be (z^2 times)

$$H_0^{\mathrm{T}} J H_0\big|_{\mathcal{O}(z^2)} = \tag{6.57}$$

$$\begin{pmatrix} -2\Lambda_{S,A} & -a_{21;A,S}^{(1)} c_{11}^{(1)} & \mathbf{1}_r - a_{21;A,S}^{(1)} a_{12;S,A}^{(1)} & -a_{21;A,S}^{(1)} c_{12}^{(1)} \\ c_{11}^{(1)\mathrm{T}} a_{21;A,S}^{(1)} & 0 & 0 & \mathbf{1}_{M-r} \\ \epsilon\,(\mathbf{1}_r + a_{12;S,A}^{(1)} a_{21;A,S}^{(1)}) & 0 & 0 & 0 \\ c_{12}^{(1)\mathrm{T}} a_{21;A,S}^{(1)} & \epsilon\,\mathbf{1}_{M-r} & 0 & 0 \end{pmatrix} ,$$

with

$$-2\Lambda_{S,A} \equiv a_{11}^{(1)\mathrm{T}} a_{21;A,S}^{(1)} - a_{21;A,S}^{(1)} a_{11}^{(1)} + c_{21} c_{22}^{\mathrm{T}} + \epsilon\, c_{22} c_{21}^{\mathrm{T}}\ . \tag{6.58}$$

This must be $H_0^{\mathrm{T}} J H_0\big|_{\mathcal{O}(z^2)} = J$, i.e. we have to eliminate the undesired terms, such that Eq. (6.57) becomes exactly equal to J. To compensate the surplus terms, we add the following extra term

$$H_0^{\text{extra}} = \begin{pmatrix} \mathbf{0}_r & & & \\ & \mathbf{0}_{M-r} & & \\ \Lambda_{S,A} & a^{(1)}_{21;A,S} c^{(1)}_{11} & a^{(1)}_{21;A,S} a^{(1)}_{12;S,A} & a^{(1)}_{21;A,S} c^{(1)}_{12} \\ & & & \mathbf{0}_{M-r} \end{pmatrix} . \tag{6.59}$$

Finally we obtain the finite coordinate patch

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})}(z) =$$
$$\begin{pmatrix} z^2\mathbf{1}_r + a^{(1)}_{11} z + a^{(0)}_{11} & c^{(1)}_{11} z + c^{(0)}_{11} & a^{(1)}_{12;S,A} z + a^{(0)}_{12;S,A} & c^{(1)}_{12} z + c^{(0)}_{12} \\ -c^{\mathrm{T}}_{22} z + b_{11} & z\mathbf{1}_{M-r} + d_{11} & 0 & d_{12} \\ a^{(1)}_{21;A,S} z + a^{(0)}_{21;A,S} + \Lambda_{S,A} & c_{21} + a^{(1)}_{21;A,S} c^{(1)}_{11} & \mathbf{1}_r + a^{(1)}_{21;A,S} a^{(1)}_{12;S,A} & c_{22} + a^{(1)}_{21;A,S} c^{(1)}_{12} \\ -\epsilon\, c^{\mathrm{T}}_{21} z + b_{21} & d_{21} & 0 & z\mathbf{1}_{M-r} + d_{22} \end{pmatrix} . \tag{6.60}$$

All other patches can be obtained by making use of the permutation (5.11):

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})}(z) \to P_{r'}^{-1} H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})'}(z) P_{r'} \, . \tag{6.61}$$

Since the transition functions between the different patches of the $k = 2$ semi-local vortices are rather complicated, we shall not discuss them in this paper; we limit ourselves to showing just a few simple examples below.

6.5.1 $G' = SO(4)$

As in the case of the $k = 2$ local vortices discussed in Sec. 5.4.2, at least nine patches are needed to describe the $k = 2$ semi-local vortices. They are divided into two disconnected parts as $9 = 5 + 4$ according to the $\mathbb{Z}_2$-parity. The five matrices corresponding to $Q_{\mathbb{Z}_2} = +1$ are $\{H_0^{(1,1)}, H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}, H_0^{(0,0)}\}$ and the four matrices with $Q_{\mathbb{Z}_2} = -1$ are $\{H_0^{(1,0)}, H_0^{(-1,0)}, H_0^{(0,1)}, H_0^{(0,-1)}\}$.

Let us start with the patches having $Q_{\mathbb{Z}_2} = +1$,

$$H_0^{(0,0)} = (z - z_0)\mathbf{1}_4 + D \, , \tag{6.62}$$
$$H_0^{(1,1)} = \begin{pmatrix} z^2\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix} + \begin{pmatrix} A_1 z + A_0 & C_{1S} z + C_{0S} \\ H_{1A} z + H_{0A} + \frac{1}{2}\left(H_{1A} A_1 - A_1^{\mathrm{T}} H_{1A}\right) & H_{1A} C_{1S} \end{pmatrix} ,$$

where D is an arbitrary 4×4 matrix. The other patches $\{H_0^{(1,-1)}, H_0^{(-1,1)}, H_0^{(-1,-1)}\}$ can be obtained by the permutations (5.11) of $H_0^{(1,1)}$.

Now we can clearly see the difference between the local and semi-local vortices. Let us consider the $(0,0)$-patch. The patches for the local vortices are given in Eq. (5.86) and those for the semi-local vortices in Eq. (6.62). To avoid confusion, let us denote them by $(0,0)_{l+}$ and $(0,0)_{l-}$ for the former and $(0,0)_{sl}$ for the latter. Clearly, the $(0,0)_{l+}$ and $(0,0)_{l-}$ patches are unified into the $(0,0)_{sl}$-patch when the strong condition is relaxed to the weak one.

As explained in Sec. 5.4.2, the $(0,0)_{l+}$ patch (with the $(1,1)$ and $(-1,-1)$ patches) and the $(0,0)_{l-}$-patch (with the $(-1,1)$ and $(1,-1)$ patches) correspond to two possible choices of the $\mathbb{Z}_2$-parities of the component vortices $(Q^{(1)}_{\mathbb{Z}_2}, Q^{(2)}_{\mathbb{Z}_2}) = (\pm1, \pm1)$. This reflects the fact that any product of the moduli matrices for local vortices generates automatically local vortices. It is tempting to interpret the fact that the two spaces are disconnected as meaning that the $\mathbb{Z}_2$-parity of each component vortex is conserved. However, this is not the case for the semi-local vortices. Products of moduli matrices satisfying the weak condition (1.72) do not, in general, satisfy it. The $\mathbb{Z}_2$-parity of each vortex is therefore not conserved in the semi-local case.

Let us examine the transition functions between the $(1,1)$ and $(0,0)$-patches, explicitly. Notice, that we have already observed the connectedness between them, as it was indeed present in the case of the local vortices. Our aim to express the following complicated results is completeness of the calculations. Let us write down the moduli matrices as

$$H_0^{(1,1)} = \begin{pmatrix} z^2 + a'_1 z + a'_0 & b'_1 z + b'_0 & e'_1 z + e'_0 & f'_1 z + f'_0 \\ c'_1 z + c'_0 & z^2 + d'_1 z + d'_0 & f'_1 z + f'_0 & g'_1 z + g'_0 \\ c'_1 i'_1 & i'_1 z + i'_0 - \frac{1}{2} a'_1 i'_1 + \frac{1}{2} d'_1 i'_1 & 1 + f'_1 i'_1 & g'_1 i'_1 \\ -i'_1 z - i'_0 - \frac{1}{2} a'_1 i'_1 + \frac{1}{2} d'_1 i'_1 & -b'_1 i'_1 & -e'_1 i'_1 & 1 - f'_1 i'_1 \end{pmatrix},$$
$$H_0^{(0,0)} = \begin{pmatrix} z + a_0 & b_0 & c_0 & d_0 \\ e_0 & z + f_0 & g_0 & h_0 \\ i_0 & j_0 & z + k_0 & l_0 \\ m_0 & n_0 & o_0 & z + p_0 \end{pmatrix}. \tag{6.63}$$

The transition functions are determined through a V-transformation (1.54) satisfying the relation $V^{(1,1),(0,0)} H_0^{(0,0)} = H_0^{(1,1)}$:

$$V^{(1,1),(0,0)} = \begin{pmatrix} z + \frac{1}{2} a'_1 + \frac{1}{2} d'_1 - \frac{i'_0}{i'_1} & 0 & 0 & \frac{1}{i'_1} \\ 0 & z + \frac{1}{2} a'_1 + \frac{1}{2} d'_1 - \frac{i'_0}{i'_1} & -\frac{1}{i'_1} & 0 \\ 0 & i'_1 & 0 & 0 \\ -i'_1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.64}$$

The transition functions connecting the patches $H_0^{(0,0)}$ and $H_0^{(1,1)}$ are thus given explicitly, see Eq. (D.2).

The transition functions between the $(1,-1)$ and $(0,0)$-patches can be obtained by the permutation of the above $(1,1)$-$(0,0)$ system as

$$P^{-1} H_0^{(1,1)} P = H_0^{(1,-1)}, \quad P^{-1} H_0^{(0,0)} P = \tilde{H}_0^{(0,0)}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6.65}$$

Therefore, the transition functions are easily found as

$$V^{(1,-1),(0,0)} \tilde{H}_0^{(0,0)} = H_0^{(1,-1)}, \quad V^{(1,-1),(0,0)} \equiv P^{-1} V^{(1,1),(0,0)} P. \tag{6.66}$$

The transition functions between the $(1,1)$ and $(1,-1)$-patches can be obtained by combining two transition functions given above.

Let us next show the transition functions between the patches with $\mathbb{Z}_2$-parity -1. The explicit form of the moduli matrix is given by

$$H_0^{(1,0)} = \begin{pmatrix} z^2 & & & \\ & z & & \\ & & 1 & \\ & & & z \end{pmatrix} + \begin{pmatrix} a_1 z + a_0 & b_1 z + b_0 & c_1 z + c_0 & d_1 z + d_0 \\ -e_1 z + e_0 & f_0 & 0 & g_0 \\ -e_1 i_1 & i_1 & 0 & e_1 \\ -i_1 z + i_0 & j_1 & 0 & k_0 \end{pmatrix} . \tag{6.67}$$

The $(-1,0)$-patch can be obtained by acting with the permutation matrix on the $(1,1)$-patch as follows

$$H_0^{(-1,0)} = P^{-1} H_0^{(1,0)\prime} P \,, \qquad P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6.68}$$

The transition functions between these two patches are obtained by

$$V^{(-1,0),(1,0)} H_0^{(1,0)} = H_0^{(-1,0)} \,, \tag{6.69}$$

$$V^{(-1,0),(1,0)} = \begin{pmatrix} 0 & 0 & -i_1' e_1' & 0 \\ 0 & 0 & -e_1' z + e_0' & -\frac{e_1'}{i_1'} \\ -\frac{1}{e_1' i_1'} & \frac{1}{e_1'}\left(z - \frac{e_0'}{e_1'}\right) & \left(z - \frac{e_0'}{e_1'}\right)\left(z - \frac{i_0'}{i_1'}\right) & \frac{1}{i_1'}\left(z - \frac{i_0'}{i_1'}\right) \\ 0 & -\frac{i_1'}{e_1'} & -i_1' z + i_0' & 0 \end{pmatrix} . \tag{6.70}$$

The other transition functions between all the other patches are obtained through suitable permutations.

It can be shown that the patches with $Q_{\mathbb{Z}_2} = +1$ and those with $Q_{\mathbb{Z}_2} = -1$ are indeed disconnected. Let us take the example of the two moduli matrices $H_0^{(0,0)}$ and $H_0^{(1,0)}$. Assume that there exists a V-function such that

$$V H_0^{(0,0)} = H_0^{(1,0)} \,. \tag{6.71}$$

First we observe that V is a matrix whose elements are all at most of order z. This is due to $H_0^{(0,0)}$ having the term, $z\mathbf{1}_4$ and the highest power of $VH_0^{(0,0)}$ should not exceed 2 which is the highest degree of $H_0^{(1,0)}$. We can thus determine the linear term in z of V

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} . \tag{6.72}$$

Furthermore, let us focus on the linear terms of z in Eq. (6.71), i.e.,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} D + \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ -e_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i_1 & 0 & 0 & 1 \end{pmatrix} . \tag{6.73}$$

By comparison of the third row of both sides, we conclude that $(v_{31}, v_{32}, v_{33}, v_{34}) = (0, 0, 0, 0$. However, $\det V = 0$ does not satisfy the requirement $V \in SO(4, \mathbb{C})$: hence these two patches are disconnected.

6.6 The $k = 1$ semi-local vortex for $G' = SO(2M + 1)$

The result of the index theorem (see Chap. 7) yields that the real dimension is $2k(2M + 1)^2$ for the moduli space in $SO(2M + 1)$. Following the technology explained in Sec. 6.5, it is straightforward to extend the results to the case of $G' = SO(2M + 1)$. The moduli matrix for $k = 1$ in the $(\overbrace{1, \cdots, 1}^{r}, \overbrace{0, \cdots, 0}^{M-r})$-patch is the most general semi-local moduli matrix and is given by

$$H_0^{(\overbrace{1,\cdots,1}^{r}, \overbrace{0,\cdots,0}^{M-r})}(z) =$$
$$\begin{pmatrix} z^2\mathbf{1}_r + a_{11}^{(1)}z + a_{11}^{(0)} & c_{11}^{(1)}z + c_{11}^{(0)} & a_{12;S}^{(1)}z + a_{12;S}^{(0)} & c_{12}^{(1)}z + c_{12}^{(0)} & e_{15}^{(1)}z + e_{15}^{(0)} \\ -c_{22}^{\mathrm{T}}z + b_{11} & z\mathbf{1}_{M-r} + d_{11} & 0 & d_{12} & e_{25} \\ a_{21;A}^{(1)}z + a_{21;A}^{(0)} + \Lambda_S & c_{21} + a_{21;A}^{(1)}c_{11}^{(1)} & \mathbf{1}_r + a_{21;A}^{(1)}a_{12;S}^{(1)} & c_{22} + a_{21;A}^{(1)}c_{12}^{(1)} & e_{35} + a_{21;A}^{(1)}e_{15}^{(1)} \\ -c_{21}^{\mathrm{T}}z + b_{21} & d_{21} & 0 & z\mathbf{1}_{M-r} + d_{22} & e_{45} \\ -e_{35}^{\mathrm{T}}z + e_{31}^{\mathrm{T}} & e_{32}^{\mathrm{T}} & 0 & e_{34}^{\mathrm{T}} & z + e_{55} \end{pmatrix} \tag{6.74}$$

where we have defined

$$-2\Lambda_S \equiv a_{11}^{(1)\mathrm{T}}a_{21;A}^{(1)} - a_{21;A}^{(1)}a_{11}^{(1)} + c_{21}c_{22}^{\mathrm{T}} + c_{22}c_{21}^{\mathrm{T}} + e_{35}e_{35}^{\mathrm{T}} \,. \tag{6.75}$$

6.6.1 $G' = SO(3)$

For $G' = SO(3), k = 1$ there are 3 patches, *viz.* (1), (-1), (0). The moduli matrix for the (0)-patch is simply

$$H_0^{(0)} = z\mathbf{1}_3 + A \,, \tag{6.76}$$

where it is noteworthy to remark that the color+flavor symmetry is unbroken.

The moduli matrix for the (1)-patch is

$$H_0^{(1)} = \begin{pmatrix} z^2 + z_1 z + z_2 & a + fz & c + bz \\ -\frac{d^2}{2} & 1 & -d \\ e + dz & 0 & z - z_3 \end{pmatrix} \,, \tag{6.77}$$

while the moduli matrix for the (-1)-patch is simply obtained by the permutation

$$H_0^{(-1)} = PH_0^{(1)}P^{-1} \,, \qquad \text{with } P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{6.78}$$

The patches (-1) and (1) are connected by a V-transformation given by

$$H'^{(1)}_0 = V^{(1),(-1)} H^{(-1)}_0 \;, \quad V^{(1),(-1)} = \begin{pmatrix} \frac{(e'+d'z')^2}{d'^2} & -\frac{2}{d'^2} & -\frac{2(e'+d'z')}{d'^2} \\ -\frac{d'^2}{2} & 0 & 0 \\ e'+d'z' & 0 & -1 \end{pmatrix} \;, \tag{6.79}$$

and the transition functions can be found in App. D. The mass center of the system can be identified by taking the coefficient of the z^2 term of $\det H_0$. It is given by: $C.M. = -z_1' + z_3' + b'd' + d'^2 f'/2 = -z_1 + z_3 + bd + d^2 f/2$, which has a form that is invariant under the change of patch.

The patches (1) and (0) are disconnected. This can be seen from identifying the linear order of V

$$H^{(1)}_0 = V H'^{(0)}_0 = V\left(z\mathbf{1}_3 + A'\right) \quad \Rightarrow \quad V = z\,\mathrm{diag}(1,0,0) + V_{\mathrm{const}} \;. \tag{6.80}$$

Looking now at the linear order in z of the equation

$$\begin{pmatrix} z_1 & f & b \\ 0 & 0 & 0 \\ d & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A' + \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix} \;, \tag{6.81}$$

which reveals that the second row of V has to be zero, which takes V out of $SO(3,\mathbb{C})$ and the patches are thus disconnected.

6.6.2 $G' = SO(5)$

For $SO(5)$ we have nine patches. The five having $\mathbb{Z}_2$ charge $+1$ are all connected and are described by the following moduli matrices

$$H^{(0,0)}(z) = z\mathbf{1}_5 + \begin{pmatrix} a_1' & a_2' & a_3' & a_4' & a_5' \\ b_1' & b_2' & b_3' & b_4' & b_5' \\ c_1' & c_2' & c_3' & c_4' & c_5' \\ d_1' & d_2' & d_3' & d_4' & d_5' \\ e_1' & e_2' & e_3' & e_4' & e_5' \end{pmatrix} \;, \tag{6.82}$$

$$H^{(1,1)}(z) = \tag{6.83}$$

$$\begin{pmatrix} z^2 + a_1 z + b_1 & a_2 z + b_2 & c_1 z + d_1 & c_2 z + d_2 & g_1 z + h_1 \\ a_3 z + b_3 & z^2 + a_4 z + b_4 & c_2 z + d_2 & c_3 z + d_3 & g_2 z + h_1 \\ e a_3 - \frac{i_1^2}{2} & ez + f - \frac{e(a_1 - a_4)}{2} - \frac{i_1 i_2}{2} & 1 + e c_2 & e c_3 & i_1 + e g_2 \\ -ez - f - \frac{e(a_1 - a_4)}{2} - \frac{i_1 i_2}{2} & -e a_2 - \frac{i_2^2}{2} & -e c_1 & 1 - e c_2 & i_2 - e g_1 \\ -i_1 z + j_1 & -i_2 z + j_2 & 0 & 0 & z + y \end{pmatrix} \;,$$

with the rest being permutations of the latter. The moduli matrix $(0, 0)$-patch is connected to the $(1, 1)$-patch by the following V-transformation

$$H^{(1,1)}(z) = V^{(1,1),(0,0)}(z)H^{(0,0)}(z) \ , \tag{6.84}$$

$$V^{(1,1),(0,0)} = \begin{pmatrix} z + \frac{a_1+a_4}{2} - \frac{f}{e} - \frac{i_1 i_2}{2e} & -\frac{i_2^2}{2e} & 0 & \frac{1}{e} & \frac{i_2}{e} \\ \frac{i_1^2}{2e} & z + \frac{a_1+a_4}{2} - \frac{f}{e} + \frac{i_1 i_2}{2e} & -\frac{1}{e} & 0 & -\frac{i_1}{e} \\ 0 & e & 0 & 0 & 0 \\ -e & 0 & 0 & 0 & 0 \\ -i_1 & -i_2 & 0 & 0 & 1 \end{pmatrix} , \tag{6.85}$$

where the transition functions can be found in App. D. There are four patches having $\mathbb{Z}_2$-charge -1, which are all connected. They are described by (and permutations of) the following moduli matrix

$$H^{(1,0)}(z) = \begin{pmatrix} z^2 + a_1 z + a_2 & c_1 z + c_0 & b_1 z + b_0 & d_1 z + d_0 & i_1 z + i_0 \\ f_0 - e_1 z & z + g_0 & 0 & g_1 & j_0 \\ -e_0 e_1 - \frac{j_1^2}{2} & e_0 & 1 & e_1 & j_1 \\ f_1 - e_0 z & g_2 & 0 & z + g_3 & j_2 \\ h_0 - j_1 z & h_1 & 0 & h_2 & z + k \end{pmatrix} . \tag{6.86}$$

This patch is connected to $H^{(-1,0)}$ by the following V-transformation

$$H^{(-1,0)}(z) = V^{(-1,0),(1,0)}(z)H^{(1,0)}(z) \ , \tag{6.87}$$

$$V^{(-1,0),(1,0)} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\Xi & 0 & 0 \\ 0 & \frac{j'^2_1}{\Xi} & f'_0 - e'_1 z & -\frac{2e'^2_1}{\Xi} & \frac{2e'_1 j'_1}{\Xi} \\ -\frac{2}{\Xi} & \frac{L_1(z)}{\Xi^2} & \frac{L_2(z)}{\Xi} & \frac{L_3(z)}{\Xi^2} & \frac{L_4(z)}{\Xi^2} \\ 0 & -\frac{2e'^2_0}{\Xi} & f'_1 - e'_0 z & \frac{j'^2_1}{\Xi} & \frac{2e'_0 j'_1}{\Xi} \\ 0 & \frac{2e'_0 j'_1}{\Xi} & -h'_0 + j'_1 z & \frac{2e'_1 j'_1}{\Xi} & 1 - \frac{2j'^2_1}{\Xi} \end{pmatrix} , \tag{6.88}$$

$$\Xi \equiv 2e'_0 e'_1 + j'^2_1 \ , \tag{6.89}$$

$$\frac{1}{2}L_1(z) \equiv f'_1 j'^2_1 - 2e'^2_0 \left(f'_0 - e'_1 z\right) + e'_0 j'_1 \left(j'_1 z - 2h'_0\right) \ , \tag{6.90}$$

$$L_2(z) \equiv h'^2_0 - 2h'_0 j'_1 z + 2f'_0 \left(f'_1 - e'_0 z\right) + z\left(2e'_0 e'_1 z + j'^2_1 z - 2e'_1 f'_1\right) \ , \tag{6.91}$$

$$\frac{1}{2}L_3(z) \equiv f'_0 j'^2_1 - 2e'^2_1 \left(f'_1 - e'_0 z\right) + e'_1 j'_1 \left(-2h'_0 + j'_1 z\right) \ , \tag{6.92}$$

$$\frac{1}{2}L_4(z) \equiv j'_1 \left(2e'_1 f'_1 + j'_1 \left(h'_0 - j'_1 z\right)\right) - 2e'_0 \left(e'_1 \left(h'_0 + j'_1 z\right) - f'_0 j'_1\right) \ . \tag{6.93}$$

The patches of different chiralities are indeed disconnected, as we expected from topological reasons.

6.7 Discussion

In this Chapter we have analyzed the BPS vortices appearing in $U(1) \times SO(N)$ and $U(1)\times USp(2M)$ gauge theories. It has been found that, in contrast to the vortices in $[U(1)\times SU(N)]/\mathbb{Z}_N \simeq U(N)$ models, the vortex moduli in these theories contain certain other moduli, generally known as semi-local vortices, whose profile functions are characterized by their asymptotic, power-like behavior, whereas the standard ANO vortices (including their non-Abelian counterparts found in $U(N)$ theories) have a sharp, exponential cutoff to their transverse size. This is so even with the minimal number of matter fields, sufficient for the system to have a "color-flavor-locked" Higgs phase. The difference with the unitary gauge group case, reflects the fact that, for a given dimension, the number of gauge degrees of freedom is less here, due to the fact that e.g. $SO(2M), USp(2M)$ groups constitute a strict subgroup $SU(2M)$.

The existence of these semi-local vortex moduli is related to the existence of non-trivial vacuum moduli of the system, and consequently, to the NLσM lumps which emerge in the strong gauge coupling limit of our vortices [5]. In this limit a vortex solution collapses to a vacuum configuration everywhere on the transverse plane. It defines a map of a 2-cycle onto the moduli space of vacua, and is thus characterized by non-trivial elements of $\pi_2(\mathcal{M}_{\rm vac})$. The existence of these semi-local moduli provides the vortex, even at finite coupling, with a very rich structure.

Related to semi-local vortices is the issue of the non-normalizability of some of the moduli space parameters. In the case of $U(N)$ vortices this question was solved completely [116], by using the general formula for the effective action of vortices in terms of the moduli matrix [113]. A part of this question was solved for a single vortex in SO and USp gauge theories in the lump limit [5]. Here we have refined our understanding of the non-normalizable modes, relating them as the moduli space parameters which live in a tangent bundle of the moduli space of vacua of the theory.

Recently, some non-BPS extensions of $U(N)$ vortices have been studied for the local case [120, 121] and for the semi-local case [119] with the aim of studying interactions and stability of non-BPS vortices. A non-BPS extension of the $G' = SO, USp$ cases also remains as an open problem. In connection with this, it is known that the $SO(2M)$ theory admits a non-BPS $\mathbb{Z}_2$ vortex as $\pi_1(SO(2M) \times U(1)) = \mathbb{Z} \times \mathbb{Z}_2$ [228, 229].

CHAPTER 7

The index theorem

In this Chapter we will derive the most generic index theorem for non-Abelian Yang-Mills-Higgs vortices with any gauge group that will allow the vanishing theorem to be applied and for any number of flavors that allows for the vacuum to break completely the gauge symmetry. The result is very simple and has already been stated in the previous Chapters.

7.1 The calculation

We briefly discuss the dimension of the vortex moduli space along the lines of Ref. [9], see also Refs. [230, 231]. In the following we will keep the gauge group completely generic with a single overall $U(1)$ factor i.e. $U(1) \times G'$. Writing the BPS equations ($e = g$) with linear fluctuations $\delta H, \delta\bar{A}$, we obtain

$$\bar{\mathcal{D}}\,\delta H = -i\,\delta\bar{A}\,H \ , \tag{7.1}$$

$$\mathcal{D}\,\delta\bar{A} - \bar{\mathcal{D}}\,\delta A = \frac{ie^2}{2}\mathrm{tr}\left\{\left(\delta H\,H^\dagger + H\,\delta H^\dagger\right)t^\alpha\right\}t^\alpha \ , \tag{7.2}$$

and the Gauss' law reads (with $\nu = 0$)

$$\mathrm{tr}\left[\left(\frac{2}{e^2}\mathcal{D}_\mu F^{\mu\nu} + iH(\mathcal{D}^\nu H)^\dagger - i(\mathcal{D}^\nu H)H^\dagger\right)t^\alpha\right] = 0 \ , \quad \forall\alpha \ , \tag{7.3}$$

which we use as a gauge fixing condition [9]

$$\mathcal{D}\,\delta\bar{A} + \bar{\mathcal{D}}\,\delta A = \frac{ie^2}{2}\mathrm{tr}\left\{\left(\delta H\,H^\dagger - H\,\delta H^\dagger\right)t^\alpha\right\}t^\alpha \ . \tag{7.4}$$

A comment in store is that one might wonder why the Gauss law is not already fulfilled by the fact that the solutions to the BPS equations satisfy the Euler-Lagrange equations of the system. Fixing the gauge can be done in many different ways, and instead of requiring the fluctuations to be orthogonal to the gauge orbit, it proves convenient to take a direction which corresponds to the time direction of the Gauss law. Even if there is no time dependence of the fields in question, we promote these fluctuations as normal fluctuations rendering the system better manageable. In other words, we constrain the a priori different directions of the fluctuations to obey the linearized Gauss law. This leads to the linear system

$$\bar{\mathcal{D}}\,\delta H = -i\,\delta\bar{A}\,H\;, \tag{7.5}$$

$$\mathcal{D}\,\delta\bar{A} = \frac{ie^2}{2}\mathrm{tr}\left(\delta H\,H^\dagger t^\alpha\right)t^\alpha\;. \tag{7.6}$$

First, we will introduce the following trick

$$\delta\bar{A} = 2\;\mathrm{tr}\left(\delta\bar{A}\;t^\beta\right)t^\beta\;, \tag{7.7}$$

which makes it possible to write the linear system conveniently as the following operator equation

$$\Delta\begin{pmatrix}\delta H\\ \delta\bar{A}\end{pmatrix} = 0\;, \tag{7.8}$$

with (taking $e^2 = 4$ for convenience)

$$\Delta \equiv \begin{pmatrix} i\bar{\mathcal{D}} & -2\;\mathrm{tr}\left(\,\circ\;t^\alpha\right)t^\alpha H\\ 2\;\mathrm{tr}\left(\,\circ\;H^\dagger t^\alpha\right)t^\alpha & i\mathcal{D}\end{pmatrix}\;, \tag{7.9}$$

which has the adjoint operator

$$\Delta^\dagger = \begin{pmatrix} i\mathcal{D} & 2\;\mathrm{tr}\left(\,\circ\;t^\alpha\right)t^\alpha H\\ -2\;\mathrm{tr}\left(\,\circ\;H^\dagger t^\alpha\right)t^\alpha & i\bar{\mathcal{D}}\end{pmatrix}\;. \tag{7.10}$$

Let us start with showing that the operator $\Delta^\dagger$ does not have any zero-modes indeed. That is, the starting point for our vanishing theorem is to take the complex norm $|X|^2 = \mathrm{tr}\,XX^\dagger$ of the operator on a fluctuation

$$\begin{aligned} 0 &= \int_{\mathbb{C}}\left|\Delta^\dagger\begin{pmatrix}X\\ Y\end{pmatrix}\right|^2 \\ &= \int_{\mathbb{C}}\Big[|\mathcal{D}X|^2 + |\bar{\mathcal{D}}Y|^2 + |YH|^2 + \left|2\mathrm{tr}\left(XH^\dagger t^\alpha\right)t^\alpha\right|^2 \\ &\qquad + i\mathrm{tr}\partial\left(XH^\dagger Y^\dagger\right) - i\mathrm{tr}\bar{\partial}\left(YHX^\dagger\right)\Big]\;, \end{aligned} \tag{7.11}$$

where the BPS equations have been used together with the fluctuation Y taking part of the algebra $Y = Y^\beta t^\beta$. This forces $Y = 0$. Here we assume the theory to be in the full Higgs phase. We take the fluctuations to vanish at spatial infinity ($|z| \to \infty$), thus the boundary terms can be neglected and we can think of the conditions

$$\bar{\mathcal{D}}X^\dagger = 0\;,\quad \bar{D}Y = 0\;,\quad YH = 0\;,\quad \mathrm{tr}\left(t^\alpha HX^\dagger\right) = 0\;, \tag{7.12}$$

as the BPS equations and F-term conditions for an $\mathcal{N} = 2$ ($d = 4$) theory with Y being the adjoint scalar of the vector multiplet and X being anti-chiral fields with the superpotential

$$W = \mathrm{tr}\left(YHX^\dagger\right)\;. \tag{7.13}$$

Recalling that this toy-theory is evaluated on the background configuration where H contains the scalar fields of the vortex and the gauge connections in the covariant derivative $\bar{A}$ are also external fields determined by the background vortex configuration. The vortex configuration can always be rewritten by means of the moduli matrix method yielding $H = S^{-1}H_0(z)$ which gives a holomorphic description of the field $X^\dagger \equiv \tilde{H}$ as $\tilde{H} = \tilde{H}_0 S$ with S the complexified gauge fields of the background configuration. It is now easy to show that the F-term condition yields $\mathrm{tr}\left(t^\alpha H_0(z)\tilde{H}_0(z)\right) = 0$, which in turn simplifies our problem to finding vacuum configurations of this $\mathcal{N} = 2$ theory, which has the vacuum in the Higgs phase almost everywhere. We utilize holomorphic invariants $I^i_\mp(H_0, \tilde{H}_0)$ having negative and positive $U(1)$ charges, respectively. The boundary conditions for the invariants are

$$I^i_- = 0\,, \quad I^i_+ = \mathcal{O}\left(z^{n_i\nu}\right)\,, \tag{7.14}$$

with ν being the $U(1)$ winding. The key point now is to find independent invariants with positive $U(1)$ charges which will reveal the possible existence of a non-zero $\tilde{H}_0$. However, the contrary is important here:

iff there exist no independent I^i_+, then the fluctuations $X^\dagger$ must vanish.

In our cases having $G = U(1) \times G'$ with $G' = U(N), SO(N), USp(2M)$ with a common $U(1)$ charge for all the fields it is an easy task to show the non-existence of independent holomorphic invariants and the theorem readily applies and completes the proof. We can now go on with the calculation.

Now let us calculate the following two operators $\Delta^\dagger\Delta$ and $\Delta\Delta^\dagger$

$$\Delta^\dagger\Delta = -\mathbf{1}_2\partial\bar{\partial} + \begin{pmatrix} \Gamma_1 + \frac{1}{2}B & L_1 \\ L_2 & \Gamma_2 - \frac{1}{2}B^{\mathrm{adj}} \end{pmatrix}\,, \tag{7.15}$$

$$\Delta\Delta^\dagger = -\mathbf{1}_2\partial\bar{\partial} + \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}\,, \tag{7.16}$$

where $B = F_{12} = -2[\mathcal{D}, \bar{\mathcal{D}}]$ and we have defined the following operators

$$\Gamma_1 X = -iA\bar{\partial}X - i(\bar{\partial}A)X - i\bar{A}\partial X + \bar{A}AX + 2\,\mathrm{tr}\left(XH^\dagger t^\alpha\right)t^\alpha H\,, \tag{7.17}$$

$$\Gamma_2 Y = -i\left[\bar{A}, \partial Y\right] - i\left[\partial\bar{A}, Y\right] - i\left[A, \bar{\partial}Y\right] + \left[A, \left[\bar{A}, Y\right]\right] + 2\,\mathrm{tr}\left(YHH^\dagger t^\alpha\right)t^\alpha\,, \tag{7.18}$$

$$L_1 Y = -iY\mathcal{D}H\,, \tag{7.19}$$

$$L_2 X = i2\,\mathrm{tr}\left(X\bar{\mathcal{D}}H^\dagger t^\alpha\right)t^\alpha\,, \tag{7.20}$$

and the algebra of Y has been used as well as the BPS equations.

To calculate the index of Δ we can evaluate

$$\mathcal{I} = \lim_{M^2\to 0}\mathcal{I}(M^2) = \lim_{M^2\to 0}\left[\mathrm{Tr}\left(\frac{M^2}{\Delta^\dagger\Delta + M^2}\right) - \mathrm{Tr}\left(\frac{M^2}{\Delta\Delta^\dagger + M^2}\right)\right]\,, \tag{7.21}$$

where Tr denotes a trace over states as well as over the matrices. Now as the eigenvalues of the operator $\Delta^\dagger$ are all positive definite, the index counts only the zero modes of the operator

Δ. For well localized solutions (which go to zero faster than $1/r$), the index is independent of M^2. For convenience we can evaluate the index in the limit $M^2 \to \infty$, thus we can expand and obtain

$$\mathcal{I}(M^2) = -M^2 \mathrm{Tr}\left[\frac{1}{-\partial\bar{\partial} + M^2}\begin{pmatrix}\frac{1}{2}B & L_1 \\ L_2 & -\frac{1}{2}B^{\mathrm{adj}}\end{pmatrix}\frac{1}{-\partial\bar{\partial} + M^2} + \ldots\right] , \tag{7.22}$$

where the ellipsis denote terms that vanish in taking the limit $M^2 \to \infty$. Tracing over the adjoint field strength gives zero. We can now evaluate the index as

$$\begin{aligned}\mathcal{I} &= -\lim_{M^2\to\infty} M^2 \mathrm{Tr}\int d^2x\, \frac{1}{2}\mathrm{tr}(F_{12})\left\langle x\left|\left(-\partial\bar{\partial} + M^2\right)^{-2}\right|x\right\rangle , \\ &= -\lim_{M^2\to\infty} M^2 \sum_1^{N_\mathrm{F}}\int d^2x\, \frac{N}{2\sqrt{2N}}F_{12}^0 \int \frac{d^2k}{(2\pi)^2}\frac{1}{\left(\frac{1}{4}k^2 + M^2\right)^2} , \\ &= N_\mathrm{F} N \nu ,\end{aligned} \tag{7.23}$$

where

$$\nu = -\frac{1}{2\pi\sqrt{2N}}\int d^2x\, F_{12}^0 = \frac{k}{n_0} . \tag{7.24}$$

Because of the vanishing theorem, the index gives exactly the number of (complex) zero-modes for the BPS equations for the vortex. Thus we obtain the same number of zero-modes as the number of moduli parameters in the moduli matrix formalism. Note that the result is obtained independently of the gauge group (however only valid when the vanishing theorem applies) and the impact of the group is simply encoded in ν. We also note that our result reduces to that of Ref. [9] for $U(N)$ by recalling that $\nu = k/N$ in that case.

CHAPTER 8

Kähler and hyper-Kähler quotients

In this Chapter we study non-linear σ models (NLσMs) whose target spaces are the Higgs phases of supersymmetric SO and USp gauge theories by using the Kähler and hyper-Kähler quotient constructions. We obtain the explicit Kähler potentials and develop an expansion formula to make use of the obtained potentials from which we also calculate the curvature of the manifolds. Furthermore, we identify singular submanifolds in the obtained manifolds, which are crucial for instance for the existence of fractional vortices.

8.1 Synopsis

The purpose of this Chapter is to explicitly construct the metric and its Kähler potential on the Higgs branch of $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric gauge theories with gauge groups $SO(N)$ and $USp(2M)$ or $U(1)\times SO(N)$ and $U(1)\times USp(2M)$. The vacua of $\mathcal{N}=1$ supersymmetric gauge theories are determined by the D-term condition, $D=0$, while those of $\mathcal{N}=2$ theories are determined by both the D-term and the F-term conditions, $D=F=0$. The moduli space of vacua is obtained by the space of solutions to these constraints modulo their gauge groups, $\{D=0\}/G$ and $\{D=F=0\}/G$ for $\mathcal{N}=1$ and $\mathcal{N}=2$ models, respectively. In the superfield formalism, solving the D-term condition and modding out the gauge group G, can be done simultaneously because the gauge symmetry is in fact complexified to $G^{\mathbb{C}}$. As a bonus the Kähler potentials are directly obtained in the superfield formalism. Although the D-term conditions of $SU(N)$ and $U(N)$ gauge groups can be solved in components easily, those of $SO(N)$ and $USp(2M)$ are difficult to solve. To our knowledge this has not been done yet. We use the superfield formalism to solve the D-term conditions for $SO(N)$ and $USp(2M)$ gauge groups by introducing a trick. Namely, we relax the algebra of the vector superfields V from $\mathfrak{so}(N)$ and $\mathfrak{usp}(N=2M)$ to $\mathfrak{u}(N)$ and then introduce a Lagrange multiplier to restrict the algebra of V to $\mathfrak{so}(N)$ and $\mathfrak{usp}(2M)$. We then successfully solve the superfield equations to obtain the resultant Kähler potentials.

There exists another method to obtain the moduli space of vacua, which is more familiar in the literature; it is the algebro-geometrical method in the geometric invariant theory [157], in which one prepares holomorphic gauge invariants made of the original chiral superfields and looks for algebraic constraints among them. This method has been widely used in the studies of $\mathcal{N}=1$ supersymmetric gauge theories [232, 233, 234]. See also Refs. [235, 236] for recent developments. In particular in Ref. [236], the moduli spaces of vacua of $\mathcal{N}=1$

supersymmetric $SO(N)$ and $USp(2M)$ gauge theories are found to be Calabi-Yau cones over certain weighted projective spaces. According to us, a weak point of the geometric invariant theory is that one has to solve algebraic constraints among invariants in order to calculate geometric quantities such as the metric and the curvature etc.

Compared with this situation our method provides the Kähler potentials directly. We rewrite them in terms of holomorphic gauge invariants. Furthermore, we calculate the metrics and the curvatures by expanding the Kähler potentials. We confirm that a singularity appears in the moduli space of vacua when the gauge symmetry is partially recovered, as expected. We then study the case of $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ gauge theories. Finally, we calculate $SO(N)$ and $USp(2M)$ hyper-Kähler quotients and obtain their Kähler potentials explicitly. Although only the lowest dimensional case $USp(2) \simeq SU(2)$ has been known so far [58], the higher dimensional cases are new.

We find explicitly the Kähler quotients for both the $\mathcal{N} = 1$ and some of the $\mathcal{N} = 2$ theories with SO and USp gauge groups, however, only at the classical level. For the $\mathcal{N} = 2$ case we are in good shape due to the well-known non-renormalization theorem on the Higgs branch by Argyres-Plesser-Seiberg [49], which leaves the results of the metric and Kähler potential quantum mechanically exact. The situation is not quite so good in the $\mathcal{N} = 1$ case. Quantum corrections should be considered, except in the compact directions of the Nambu-Goldstone modes (up to overall constants: pion decay constants) which is indeed consistent with the low-energy theorem of Nambu-Goldstone modes. Along the non-compact directions parametrized by quasi-Nambu-Goldstone modes, the corrections are out of control and can render rather large. All in all, the total Kähler potential is correct only (semi-)classically for the $\mathcal{N} = 1$ case and it will take the form

$$K = f(I_1, I_2, \ldots) \,, \tag{8.1}$$

with I_a being $G^{\mathbb{C}}$ invariants and f some function. In the case of $\mathcal{N} = 1, U(N)$ theories, some quantum corrections have been considered in the literature [237, 238, 239, 240, 241, 242]. To this end, we emphasize that the metric and Kähler potential was until now unknown, even classically and the first step has been taken, which of course leaves the quantum corrections as an important and interesting future calculation to grasp.

8.2 The $SO(N)$ and $USp(2M)$ Kähler quotients

The Kähler potential for an $SO(N)$ or a $USp(2M)$ gauge theory is given by

$$K_{SO,USp} = \mathrm{Tr}\left[QQ^\dagger e^{-V'}\right] \,, \tag{8.2}$$

where V' takes a value in the $\mathfrak{so}(N)$ or $\mathfrak{usp}(2M)$ algebra. The D-flatness conditions in the Wess-Zumino gauge are

$$D^A = \mathrm{Tr}_{\mathrm{F}}\left(Q_{\mathrm{wz}}^\dagger T^A Q_{\mathrm{wz}}\right) = 0 \,, \tag{8.3}$$

with T_A being the generators in the Lie algebra of SO or USp.

Instead of solving these equations explicitly, we will here discuss the breaking pattern of the gauge symmetry and the flat directions. For this we will use both the gauge and the global symmetry as usually done. The vacuum expectation value of $Q_{\rm wz}^{SO}$ in the case of $SO(N)$ can be put on the diagonal form after fixing both the local and the global symmetries as [233]

$$Q_{\rm wz}^{SO(N)} = \left(A_{N\times N},\ \mathbf{0}_{N\times(N_{\rm F}-N)}\right)\ , \qquad A_{N\times N} = {\rm diag}(a_1, a_2, \cdots, a_N)\ , \tag{8.4}$$

where we have taken a normal basis for the $SO(N)$ group, namely $g^{\rm T} g = \mathbf{1}_N$. Here all the parameters a_i are taken to be real and positive, which indeed parametrize the flat directions of the Higgs branch. In generic points of the moduli space of vacua with non-degenerate a_i, the gauge symmetry is completely broken and the flavor symmetry $U(N_{\rm F})$ is broken to $U(N_{\rm F} - N)$. The moduli space of vacua can be locally written in generic points as

$$\mathcal{M}_{SO(N)} \simeq \mathbb{R}^N_{>0} \times \frac{U(N_{\rm F})}{U(N_{\rm F}-N)\times(\mathbb{Z}_2)^{N-1}}\ . \tag{8.5}$$

Here the discrete unbroken group $(\mathbb{Z}_2)^{N-1}$ has elements of N-by-N diagonal matrices in the $SO(N)$ group elements acting from the left, which have an even number of -1 elements with the rest 1, in addition to the same matrices embedded into the $U(N_{\rm F})$ group acting from the right. We see that the space is of cohomogeneity N, of which the isometry is $U(N_{\rm F})$ and the isotropy at generic points is $U(N_{\rm F} - N)$. The coordinates of the coset space $U(N_{\rm F})/U(N_{\rm F}-N)$ correspond to Nambu-Goldstone (NG) modes of the broken flavor symmetry, whereas the coordinates $\{a_i\}$ of the flat directions $\mathbb{R}^N_{>0}$ correspond to the so-called "quasi-Nambu-Goldstone" modes [221, 222, 223, 225]. The quasi-NG modes do not correspond to a symmetry breaking but are ensured by supersymmetry. In general, the unbroken flavor symmetry, namely the isotropy of the space, changes from point to point depending on the values of the parameters (the quasi-NG modes) a_i's. When two parameters coincide, $a_i = a_j, (i \neq j)$, a color-flavor locking $SO(2)$ symmetry emerges. In such degenerate subspace on the manifold, the above coset space attached to $\mathbb{R}^N_{>0}$ shrinks to one with a smaller dimension;[1]

$$\mathcal{M}_{SO(N)} \sim \mathbb{R}^{N+1}_{>0} \ltimes \frac{U(N_{\rm F})}{U(N_{\rm F}-N)\times SO(2)\times(\mathbb{Z}_2)^{N-2}}\ . \tag{8.6}$$

In general, when n_i ($i = 1, 2, \cdots$, and $\sum_i n_i \leq N$) parameters among a_i coincide, the symmetry structure of the moduli space of vacua becomes

$$\mathcal{M}_{SO(N)} \sim \mathbb{R}^{N+\sum_i \frac{1}{2} n_i(n_i-1)}_{>0} \ltimes \frac{U(N_{\rm F})}{U(N_{\rm F}-N)\times \prod_i SO(n_i)\times(\mathbb{Z}_2)^{N-1-\sum_i(n_i-1)}}\ . \tag{8.7}$$

The most symmetric vacuum, when all parameters coincide, is realized as

$$\mathcal{M}_{SO(N)} \sim \mathbb{R}^{\frac{1}{2}N(N+1)}_{>0} \ltimes \frac{U(N_{\rm F})}{U(N_{\rm F}-N)\times SO(N)}\ . \tag{8.8}$$

[1] Some quasi-NG modes change to NG modes reflecting further symmetry breaking. This change of quasi-NG and NG modes was pointed out in Refs. [224, 226]. This was also observed in the moduli space of domain walls [243] and of non-Abelian vortices [101], where quasi-NG modes correspond to the positions of solitons. Here the notation "$\ltimes$" is used for a local structure of the bundle $F \ltimes B$ with a fiber F and a base space B. This is not globally true; once some values of $\mathbb{R}^{\#}_{>0}$ change, the coset space changes in general.

This breaking pattern of the flavor symmetry is the one of non-supersymmetric $SO(N)$ QCD [228, 229]. The unbroken flavor symmetry in non-supersymmetric QCD is in general further broken down as in Eq. (8.5) or (8.7) in supersymmetric QCD.

No singularities appear in the moduli space even when the parameters coincide unless they vanish. The existence of the quasi-NG modes is strongly related to the emergence of the Coulomb phase. When one of the a_i's vanishes, the NG part becomes $U(N_{\rm F})/U(N_{\rm F}-N+1)$ but the gauge symmetry is still completely broken. Accordingly, no singularities appear. However, when any two of the a_i's vanish, an $SO(2)$ subgroup of the gauge symmetry is recovered and the NG part becomes $U(N_{\rm F})/U(N_{\rm F}-N+2)$. (One expects a singularity on the manifold in the limit of two vanishing a_i's). Thus, in the Higgs phase with completely broken gauge symmetry, the rank of $Q_{\rm wz}$ has to be greater than $N-2$. In this Chapter, we consider this latter case, the models with $N_{\rm F} \geq N-1$.

For the $USp(2M)$ case it is known that the flat directions are parametrized by [234, 50]

$$Q_{\rm wz}^{USp(2M)} = \mathbf{1}_2 \otimes \left(A_{M\times M},\ \mathbf{0}_{M\times(M_{\rm F}-M)}\right) , \tag{8.9}$$

where the number of flavors is even $N_{\rm F} = 2M_{\rm F}$. Even in generic points with non-degenerate $\{a_i\}$, the color-flavor symmetries $USp(2)^M \simeq SU(2)^M$ exist in the vacuum. Therefore, the moduli space of vacua can be locally written in generic points as

$$\mathcal{M}_{USp(2M)} \simeq \mathbb{R}^M_{>0} \times \frac{U(N_{\rm F})}{U(N_{\rm F}-2M)\times USp(2)^M} , \tag{8.10}$$

except for submanifolds where the coset space shrinks. The resulting space is of cohomogeneity M. Again, when n_i ($i=1,2,\cdots$, and $\sum_i n_i \leq M$) parameters among a_i coincide, the symmetry structure becomes

$$\mathcal{M}_{USp(2M)} \sim \mathbb{R}^{M+2\sum_i n_i(n_i-1)}_{>0} \ltimes \frac{U(N_{\rm F})}{U(N_{\rm F}-2M)\times USp(2)^{M-\sum_i n_i}\times \prod_i USp(2n_i)} . \tag{8.11}$$

The most symmetric vacuum, when all parameters coincide, is realized as

$$\mathcal{M}_{USp(2M)} \sim \mathbb{R}^{M(2M-1)}_{>0} \ltimes \frac{U(N_{\rm F})}{U(N_{\rm F}-2M)\times USp(2M)} , \tag{8.12}$$

whose breaking pattern is the one of non-supersymmetric $USp(2M)$ QCD. There are no singularities unless one of the parameters a_i vanishes. In the case of $USp(2M)$ the complete broken gauge symmetry needs $M_{\rm F} \geq M$.

Next we explicitly construct the Kähler potentials from the moduli space of vacua. The D-flatness conditions (8.3), however, are rather difficult to solve.[2] Without taking the Wess-Zumino gauge, we can eliminate the superfield V' directly within the superfield formalism by using a trick. To this end we note that V' satisfies $\det(e^{-V'}) = 1$ and

$$V'^{\rm T} J + J V' = 0 \quad \leftrightarrow \quad e^{-V'^{\rm T}} J e^{-V'} = J . \tag{8.13}$$

[2] To our knowledge the D-flatness conditions are not solved in the case of an SO or a USp, $\mathcal{N}=1$ supersymmetric gauge theory.

Here the matrix J is the invariant tensor of the SO or USp group, $g^{\mathrm{T}}Jg = J$ with $g \in SO(N),\ USp(2M)$, satisfying

$$J^{\mathrm{T}} = \epsilon J\ , \quad J^\dagger J = \mathbf{1}_N\ , \qquad \epsilon = \begin{cases} +1 & \text{for} \quad SO(N)\ , \\ -1 & \text{for} \quad USp(N=2M)\ . \end{cases} \tag{8.14}$$

We can choose the form of the invariant tensor J as[3]

$$J_M^{\pm} \equiv \begin{pmatrix} \mathbf{0}_M & \mathbf{1}_M \\ \pm\mathbf{1}_M & \mathbf{0}_M \end{pmatrix}\ , \qquad J_{M,\mathrm{odd}} \equiv \begin{pmatrix} J_M^+ & \vec{0}^{\mathrm{T}} \\ \vec{0} & 1 \end{pmatrix}\ , \tag{8.15}$$

where the last tensor is for the $SO(N = 2M+1)$ case. We will use these conventions throughout the paper unless otherwise stated.

We are now ready to eliminate V' using the following trick. Let us first consider V' taking a value in a larger algebra, namely $\mathfrak{u}(N)$ and then introduce an N-by-N matrix of Lagrange multipliers[4] λ to restrict V' to take a value in the $\mathfrak{so}(N)$ or the $\mathfrak{usp}(N=2M)$ subalgebra:

$$K_{SO,USp} = \mathrm{Tr}\left[QQ^\dagger e^{-V'} + \lambda\left(e^{-V'^{\mathrm{T}}}Je^{-V'} - J\right)\right]\ , \tag{8.16}$$

where Q are N_{F} chiral superfields as earlier and V' is a vector superfield of $U(N)$. The added term breaks the complexified gauge transformation to $SO(N)$, $USp(2M)$ and the equation of motion for λ gives the constraint (8.13) which reduces the Kähler potential (8.16) back to (8.2). Instead, we will take another path and eliminate V'. The equation of motion for V' takes the form

$$QQ^\dagger e^{-V'} + \left(\lambda + \epsilon\lambda^{\mathrm{T}}\right)J = 0\ , \tag{8.17}$$

where we have used (8.13). Combining (8.17) with its transpose: $e^{-V'^{\mathrm{T}}}Q^*Q^{\mathrm{T}} + J(\lambda + \epsilon\lambda^{\mathrm{T}}) = 0$, then λ can be eliminated:

$$QQ^\dagger e^{-V'} = e^{V'}J^\dagger Q^*Q^{\mathrm{T}}J\ . \tag{8.18}$$

Furthermore, in order to make the equation Hermitian, we multiply by $\sqrt{QQ^\dagger}e^{-V'}$ from the left and by $\sqrt{QQ^\dagger}$ from the right:

$$X^2 = \left(Q^{\mathrm{T}}J\sqrt{QQ^\dagger}\right)^\dagger\left(Q^{\mathrm{T}}J\sqrt{QQ^\dagger}\right)\ , \qquad X \equiv \sqrt{QQ^\dagger}e^{-V'}\sqrt{QQ^\dagger}\ . \tag{8.19}$$

This equation uniquely gives a positive definite matrix X, by means of its square root. We can thus uniquely obtain V' from this X, if and only if the holomorphic invariants $M \equiv Q^{\mathrm{T}}JQ$ satisfy $\mathrm{rank}\,M > N-2$, that is, if and only if the vacuum is in the full Higgs phase. See App. F for a uniqueness proof, in the case of $\mathrm{rank}\,M = N-1$. It is

[3] Two arbitrary choices of the invariant tensor are related by an appropriate unitary transformation u : $J' = u^{\mathrm{T}}J\,u$. Correspondingly, the elements of the gauge group for different choices of the invariant tensor are related by $g' = u^\dagger g u$. See App. E.

[4] Hermiticity of λ is defined so that $\lambda e^{-V'^{\mathrm{T}}}J$ is a vector superfield, that is, $\lambda^\dagger = e^{V'^{\mathrm{T}}}J\lambda\,e^{-V'^{\mathrm{T}}}J$.

possible to switch to $Q_{\rm wz}$ from Q by the complexified gauge transformation $Q_{\rm wz} = u'^{-1}Q$ with $u'u'^\dagger = e^{V'}$. Without using an explicit solution for V', we obtain the Kähler potential of the NLσM

$$K_{SO,USp} = {\rm Tr}X = {\rm Tr}\sqrt{\left(Q^{\rm T}J\sqrt{QQ^\dagger}\right)^\dagger \left(Q^{\rm T}J\sqrt{QQ^\dagger}\right)}\ . \tag{8.20}$$

Thus we have obtained the explicit Kähler potentials.

Now we can naturally switch to another expression for this NLσM in terms of the holomorphic gauge invariants. With the help of ${\rm Tr}\sqrt{AA^\dagger} = {\rm Tr}_{\rm F}\sqrt{A^\dagger A}$, one can rewrite the Kähler potential (8.20) as

$$K_{SO,USp} = {\rm Tr}_{\rm F}\sqrt{MM^\dagger}\ , \qquad M^{\rm T} = \epsilon M\ , \tag{8.21}$$

where M is nothing but the holomorphic invariants of the gauge symmetry

$$M \equiv Q^{\rm T}JQ\ , \qquad B^{\langle A\rangle} \equiv \det Q^{\langle A\rangle}\ . \tag{8.22}$$

The first one is the "mesonic" invariant while the second is the "baryonic" one which appears for $N_{\rm F} \geq N$. The two kinds of invariants should be subject to constraints in order to correctly describe the NLσM. There are relations between the mesons and the baryons:

$$SO(N) : \det(J)\ B^{\langle A\rangle}B^{\langle B\rangle} = {\rm det}M^{\langle A\rangle\langle B\rangle}, \tag{8.23}$$

$$USp(2M) : {\rm Pf}(J)\ B^{\langle A\rangle} = {\rm Pf}\, M^{\langle A\rangle\langle A\rangle}. \tag{8.24}$$

where the N-by-N matrix $M^{\langle A\rangle\langle B\rangle}$ is a minor matrix defined by $\left(M^{\langle A\rangle\langle B\rangle}\right)^{ij} = M^{A_iB_j}$. The Plücker relations among the baryonic invariants $B^{\langle A\rangle}$ are derived from the above relation. Actually, from the invariants M and $B^{\langle A\rangle}$ with the constraints we can reconstruct Q modulo the complexified gauge symmetry as follows. By using an algorithm similar to the Cholesky decomposition of an Hermitian matrix, we can show that

An arbitrary n-by-n (anti-)symmetric complex matrix X can always be decomposed as $X = p^{\rm T}Jp$ with a ${\rm rank}(X)$-by-n matrix p. (8.25)

See App. E.2 for a proof of this statement. In the USp case, with a decomposition of the meson M, we can completely reconstruct Q modulo $USp(2M)^{\mathbb C}$ transformations. This fact corresponds to the fact that there are no independent baryons $B^{\langle A\rangle}$ in this $USp(2M)$ theory and only the meson fields describe the full Higgs phase

$$\mathcal{M}_{USp} = \left\{M\,|\, M \in N_{\rm F}\text{-by-}N_{\rm F}\ {\rm matrix}, \quad M^{\rm T} = -M, \quad {\rm rank}\, M = 2M\right\}\ . \tag{8.26}$$

On the contrary, in the $SO(N)$ case, a decomposition of M gives Q modulo $O(N)^{\mathbb C}$ and one finds two candidates for Q since $\mathbb{Z}_2 \simeq O^{\mathbb C}/SO^{\mathbb C}$ which is fixed by the sign of the baryons.[5] Therefore we have to take the degrees of freedom of the baryons into account to consider the full Higgs phase

$$\mathcal{M}_{SO} = \left\{M, B^{\langle A\rangle}\,|\, M :\ {\rm symmetric}\ N_{\rm F}\text{-by-}N_{\rm F}, {\rm Eq.}\ (8.23),\ N-1 \leq {\rm rank}\, M \leq N\right\}. \tag{8.27}$$

[5] In the case of ${\rm rank}\, M = N-1$, $g \in \mathbb{Z}_2$ acts trivially on Q as $g\,Q = Q$, although all the baryons vanish.

For large N, it is a hard task to obtain an explicit metric from the formula (8.21), since we need to calculate the eigenvalues of $MM^\dagger$. Let us, therefore, consider expanding the Kähler potential (8.21) in terms of infinitesimal coordinates around a point. Note that the meson field M for $SO(N)$, which is a symmetric matrix, can always be diagonalized by using the flavor symmetry $U(N_{\rm F})$ as

$$M_{\rm vev}^{SO} \equiv uMu^{\rm T} = {\rm diag}(\mu_1, \mu_2, \cdots, \mu_N, 0, \cdots)\ , \tag{8.28}$$

with $u \in U(N_{\rm F})$ and the parameters $\mu_i \in \mathbb{R}_{\geq 0}$ are the square roots of the eigenvalues of $MM^\dagger$. The meson field M in the $USp(2M)$ case, which is an anti-symmetric matrix, can be also diagonalized as

$$M_{\rm vev}^{USp} \equiv uMu^{\rm T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes {\rm diag}(\mu_1, \mu_2, \cdots, \mu_M, 0, \cdots)\ . \tag{8.29}$$

See App. E.2 for the proof. These vacuum configurations in both the cases, $M_{\rm vev} = M_{\rm vev}^{SO}$, $M_{\rm vev}^{USp}$, are summarized as

$$(M_{\rm vev})_{ij} = \mu_i (J)_{ij} = (J)_{ij}\mu_j\ , \tag{8.30}$$

where we take the invariant tensors as $(J)_{ij} = \delta_{ij}$ for the $SO(N)$ case, and $(J)_{ij} = \delta_{i+M_{\rm F},j} - \delta_{i,j+M_{\rm F}}$ and $\mu_{i+M_{\rm F}} \equiv \mu_i$, $(1 \leq i \leq M_{\rm F})$ in the case of $USp(N = 2M)$.

For simplicity, let us concentrate on the $SO(N)$ case with $N = N_{\rm F}$, and consider generic points of the manifold with ${\rm rank}(M_{\rm vev}) = N$, that is, $\mu_i > 0$ for all i. In this case, there are no constraints for the meson field locally, and thus, the meson field M can be treated as coordinates parametrizing the manifold locally. It is convenient to consider a small fluctuation $\phi = M - M_{\rm vev}$ around the vacua $M_{\rm vev}$ and expand the formula (8.21) with respect to ϕ. The following formula is useful to expand a function $f(X)$ of a matrix X in a trace around $X = X_0$,

$$\begin{aligned} {\rm Tr}[f(X_0 + \delta X)] &= \frac{1}{2\pi i}\oint_{\mathcal{C}} d\lambda\ f(\lambda){\rm Tr}\left[\frac{\mathbf{1}}{\lambda\mathbf{1} - X_0 - \delta X}\right] \\ &= {\rm Tr}[f(X_0)] + \sum_{n=1}^{\infty} \frac{1}{2\pi n\, i}\oint_{\mathcal{C}} d\lambda\ f'(\lambda){\rm Tr}\left[\left(\frac{\mathbf{1}}{\lambda\mathbf{1} - X_0}\delta X\right)^n\right]\ , \end{aligned} \tag{8.31}$$

where the closed path $\mathcal{C}$ surrounds all eigenvalues of $f(X)$ on the real positive axis but no singularities of $f(\lambda)$. We set $f(\lambda) = \sqrt{\lambda}$ and

$$X = MM^\dagger\ , \quad X_0 = {\rm diag}(\mu_1^2, \cdots, \mu_N^2)\ , \quad \delta X = M_{\rm vev}\phi^\dagger + \phi M_{\rm vev}^\dagger + \phi\phi^\dagger\ . \tag{8.32}$$

Since $f(\lambda) = \sqrt{\lambda}$ has a branch point at the origin, the eigenvalues μ_i cannot be zero in this formula. To proceed the calculation, we need to perform the integrations

$$A_n(\mu_1, \cdots, \mu_n) \equiv \frac{1}{2\pi i}\oint \frac{d\lambda}{\sqrt{\lambda}} \prod_{i=1}^{n} \frac{1}{\lambda - \mu_i^2}\ . \tag{8.33}$$

The results of the integrations can be expressed in terms of the elementary symmetric polynomials, $C^{(m)}_{k_1k_2\cdots k_n}, (m \le n)$ defined by

$$\prod_{i=1}^{n}(t+\mu_{k_i}) = \sum_{m=0}^{n} C^{(m)}_{k_1\cdots k_n} t^{n-m} , \quad P_{k_1k_2\cdots k_n} \equiv \prod_{m>n}(\mu_{k_m}+\mu_{k_n}) , \tag{8.34}$$

where we also use a symmetric polynomial $P_{k_1\cdots k_n}$. The first few integrations give

$$\begin{aligned} A_1(\mu_1) &= \frac{1}{\mu_1} , \quad A_2(\mu_1,\mu_2) = -\frac{1}{\mu_1\mu_2(\mu_1+\mu_2)} , \\ A_3(\mu_1,\mu_2,\mu_3) &= \frac{C^{(1)}_{123}}{C^{(3)}_{123}P_{123}} = \frac{\mu_1+\mu_2+\mu_3}{\mu_1\mu_2\mu_3(\mu_1+\mu_2)(\mu_2+\mu_3)(\mu_3+\mu_1)} , \\ A_4(\mu_1,\mu_2,\mu_3,\mu_4) &= -\frac{C^{(1)}_{1234}C^{(2)}_{1234}-C^{(3)}_{1234}}{C^{(4)}_{1234}P_{1234}} . \end{aligned} \tag{8.35}$$

After this preparation, we obtain the first few terms of the expansion of the Kähler potential as

$$\begin{aligned} K_{SO} \;=\; & \frac{1}{2}\sum_{i,j}\frac{\phi_{ij}\phi^\dagger_{ji}}{\mu_i+\mu_j} \\ & -\frac{1}{2}\sum_{i,j,k}\frac{\mu_i\,\phi_{ij}\phi^\dagger_{jk}\phi_{ki}}{(\mu_i+\mu_j)(\mu_j+\mu_k)(\mu_k+\mu_i)} + \text{c.c.} \\ & +\frac{1}{2}\sum_{i,j,k,l}\frac{\mu_j\mu_k C^{(1)}_{ijkl}}{P_{ijkl}}\phi_{ij}\phi_{jk}\phi_{kl}\phi^\dagger_{li} + \text{c.c.} \\ & +\frac{1}{2}\sum_{i,j,k,l}\frac{\mu_j\mu_l C^{(1)}_{ijkl}}{P_{ijkl}}\phi_{ij}\phi_{jk}\phi^\dagger_{kl}\phi^\dagger_{li} - \frac{1}{4}\sum_{i,j,k,l}\frac{C^{(3)}_{ijkl}}{P_{ijkl}}\phi_{ij}\phi^\dagger_{jk}\phi_{kl}\phi^\dagger_{li} \\ & + \text{Kähler trf.} + \mathcal{O}(\phi^5) . \end{aligned} \tag{8.36}$$

A coordinate singularity emerges in the limit $\mu_i \to 0$ since the expansion formula (8.31) is not applicable for $\mu_i = 0$. The above result gives enough information to calculate the scalar curvature R of the manifold at $M = M_{\rm vev}$ in the $SO(N)$ case, with a Kähler metric $g_{I\bar{J}}$

$$\begin{aligned} R|_{\phi=0} &= -2g^{I\bar{J}}\partial_I\partial_{\bar{J}}\log\det g\Big|_{\phi=0} \\ &= 2\sum_{i>j}\left(\frac{1}{\mu_i+\mu_j}+\sum_k\frac{\mu_k}{(\mu_k+\mu_i)(\mu_k+\mu_j)}\right) > 0 , \end{aligned} \tag{8.37}$$

where the indices $I, \bar{J}$ label the components as $\phi^I = \phi_{ij}, (i \ge j)$. This result shows that the coordinate singularity with $\mathrm{rank}(M_{\rm vev}) = N-1$ can be removed by taking appropriate coordinates and, on the other hand, the submanifold with $\mathrm{rank}(M_{\rm vev}) < N-1$ is a curvature singularity of the manifold. That is, the curvature singularity lies in the region corresponding to the Coulomb phase of the original gauge theory, as we expected.

The expansion of the Kähler potential in the $USp(2M)$ case, we obtain the result (8.36) with the substitution $\phi \to \phi J^\dagger$, $\phi^\dagger \to J\phi^\dagger$ and the curvature obtained using this expanded potential reads

$$R|_{\phi=0} = 4\sum_{i>j}^{M}\left(\frac{1}{\mu_i+\mu_j} + \sum_{k}^{M}\frac{4\mu_k}{(\mu_k+\mu_i)(\mu_k+\mu_j)}\right) > 0 \,. \tag{8.38}$$

This result shows that the submanifold with $\mathrm{rank}(M_{\mathrm{vev}}) < 2(M-1)$ is a curvature singularity of the manifold. This expansion, however, does not reveal the singularity appearing at $\mathrm{rank}(M_{\mathrm{vev}}) = 2(M-1)$. To detect this singularity, we consider a deformation of the Kähler potential

$$K_{USp,\mathrm{deformed}} = \mathrm{Tr}\,\sqrt{MM^\dagger + \varepsilon^2}\,, \tag{8.39}$$

and make a similar expansion (see App. G). Taking now only one eigenvalue, say $\mu_1 \to 0$ we find a term in the scalar curvature

$$\lim_{\mu_1\to 0} R|_{\phi=0} \supset \frac{2}{\varepsilon}\,, \tag{8.40}$$

which shows the presence of a singularity for one vanishing eigenvalue, that is corresponding to an unbroken $USp(2) \simeq SU(2)$ symmetry.

8.3 The $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ Kähler quotients

Next, we would like to consider a Kähler quotient with gauging an overall $U(1)$ phase in addition to the $SO(N)$ or $USp(2M)$ gauge symmetry. We turn on the FI D-term associated with the additional $U(1)$ gauge group. The Kähler potential can be written as

$$K_{U(1)\times(SO,USp)} = \mathrm{Tr}\left[QQ^\dagger e^{-V'}e^{-V_e} + \lambda\left(e^{-V'^{\mathrm{T}}}Je^{-V'} - J\right)\right] + \xi V_e\,, \tag{8.41}$$

where V_e is the vector multiplet of the additional $U(1)$ gauge field. We have already solved the $SO(N)$ and $USp(2M)$ part in the previous Section, so the Kähler potential can be rewritten as

$$K_{U(1)\times(SO,USp)} = \mathrm{Tr}\left[\sqrt{MM^\dagger}\right]e^{-V_e} + \xi V_e\,. \tag{8.42}$$

The equation of motion for V_e can be solved by $V_e = \log\left[\mathrm{Tr}\left(\sqrt{MM^\dagger}\right)/\xi\right]$. Plugging this into the Kähler potential, we obtain

$$K_{U(1)\times(SO,USp)} = \xi\log\left[\mathrm{Tr}\left(\sqrt{MM^\dagger}\right)\right]\,, \qquad M \equiv Q^{\mathrm{T}}JQ\,. \tag{8.43}$$

In the case of $N = N_{\rm F}$, we can expand the Kähler potential around a point $M = M_{\rm vev}$ by using the same method as in Sec.8.2,

$$\begin{aligned} K_{U(1)\times(SO,USp)} &= \frac{\xi}{2\sum_{k=1}^{N}\mu_k}\left(\sum_{i,j}^{N}\frac{\phi_{ij}(\phi_{ij})^\dagger}{\mu_i+\mu_j} - \frac{1}{2\sum_{k=1}^{N}\mu_k}\left|\sum_{i=1}^{N}(J^\dagger\phi)_{ii}\right|^2\right) \\ &\quad + \text{Kähler trf.} + \mathcal{O}(\phi^3)\ . \end{aligned} \tag{8.44}$$

Here we can confirm that the mode $\phi \propto M_{\rm vev}$ corresponding to $U(1)^{\mathbb{C}}$ is not effective in this Kähler potential. Therefore, with the constraint Tr $[\phi J^\dagger] = 0$, we can write the Kähler potential to fourth order as

$$\begin{aligned} K_{U(1)\times(SO,USp)} = \frac{\xi}{\sum_{k=1}^{N}\mu_k}\Bigg[K_{SO,USp} &- \frac{1}{8\sum_{l=1}^{N}\mu_l}\left|\sum_{i,j}\frac{\phi_{ij}\phi_{ji}^\dagger}{\mu_i+\mu_j}\right|^2 \\ &- \frac{1}{16\sum_{l=1}^{N}\mu_l}\left|\sum_{i,j}\frac{(\phi J^\dagger)_{ij}(\phi J^\dagger)_{ji}}{\mu_i+\mu_j}\right|^2\Bigg] \\ &+ \text{ Kähler trf.} + \mathcal{O}(\phi^5)\ . \end{aligned} \tag{8.45}$$

from which we obtain the curvatures as

$$\xi R_{U(1)\times(SO,USp)} = R_{(SO,USp)}\sum_{i=1}^{N}\mu_i + 2\hat{N}_\epsilon(\hat{N}_\epsilon+1)\ , \tag{8.46}$$

where $\hat{N}_\epsilon$ is the complex dimension of the manifold

$$\hat{N}_\epsilon \equiv \dim_{\mathbb{C}}\mathcal{M}^{\rm vacuum}_{U(1)\times(SO,USp)} = \frac{N(N+\epsilon)}{2} - 1\ , \qquad \epsilon = \begin{cases} +1 & \text{for} \quad SO\ , \\ -1 & \text{for} \quad USp\ . \end{cases} \tag{8.47}$$

A typical property of these theories is the existence of curvature singularities of the Kähler manifold. Since the Coulomb phase attached to the Higgs phase in the original gauge theory is strongly related to a singularity, the curvature singularity with $0 < \text{rank}(M) < N-1$ still survives after the $U(1)$ gauging for the case of $N \geq 3$, while gauging $U(1)$ in the $SU(N)$ case removes the singularity.

8.4 Examples

8.4.1 The $SO(2)$ quotient (SQED) and the $U(1)\times SO(2)$ quotient

The first example is $SO(2)$ with $N_{\rm F} = 1$. We have a complexified gauge symmetry $SO(2)^{\mathbb{C}}$, so the corresponding target space is

$$\mathcal{M}^{SO(2)}_{N_{\rm F}=1} = Q/\sim\ , \quad Q \sim g'Q\ , \quad g' \in SO(2)^{\mathbb{C}}\ , \tag{8.48}$$

where $Q = (Q_+,\ Q_-)^{\mathrm{T}}$. In general, matrices in $SO(2)^{\mathbb{C}}$ can be expressed as

$$g' = \begin{pmatrix} v' & 0 \\ 0 & 1/v' \end{pmatrix} , \quad v' \in \mathbb{C}^* . \tag{8.49}$$

This simply shows the fact that $SO(2) \simeq U(1)$ under which Q_+ has charge $+1$ while Q_- has charge -1. This is nothing else than supersymmetric QED. The target space apparently seems to be a weighted complex projective space which is not a Hausdorff space

$$\mathcal{M}_{N_{\mathrm{F}}=1}^{SO(2)} = W\mathbb{C}P^1_{(1,-1)} . \tag{8.50}$$

However, we have to be careful. Sick points $(Q_+, Q_-) = (Q_+, 0),\ (0, Q_-)$ for $Q_+ \neq 0$ and $Q_- \neq 0$ are forbidden by the D-term condition $|Q_+|^2 - |Q_-|^2 = 0$ in the Wess-Zumino gauge. To understand the true well-defined target space, we take the holomorphic invariant of this model to be

$$M = 2Q_+Q_- . \tag{8.51}$$

This is a good coordinate on the target space and the Kähler potential is given by

$$K_{N_{\mathrm{F}}=1}^{SO(2)} = |M| . \tag{8.52}$$

There is a conical singularity at the origin and the true target space is

$$\mathcal{M}_{N_{\mathrm{F}}=1}^{SO(2)} = \mathbb{C}/\mathbb{Z}_2 . \tag{8.53}$$

At the singularity, the gauge symmetry is restored and the vector multiplet obtains a massless field. In general, singularities in a classical moduli space lead to the appearance of some massless fields. Kähler potentials usually acquire quantum corrections and they may make such classical singular manifolds regular.

The second example is $U(1) \times SO(2)$ with $N_{\mathrm{F}} = 1$. We turn on the FI parameter ξ and we have

$$\mathcal{M}_{N_{\mathrm{F}}=1}^{U(1)\times SO(2)} = Q/\!\sim , \quad Q \sim V_e V' Q , \quad V_e \in U(1)^{\mathbb{C}} , \quad V' \in SO(2)^{\mathbb{C}} . \tag{8.54}$$

We can explicitly show that

$$g_e g' = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad v_1, v_2 \in \mathbb{C}^* . \tag{8.55}$$

Here we impose that the gauge symmetry $U(1) \times SO(2)$ is free, such that $|Q| \neq 0$. Hence, the target space is just one point.

Next, let us consider $N_{\mathrm{F}} = 2$ with the $SO(2)$ and the $U(1) \times SO(2)$ gauge groups. The scalar field is a 2-by-2 complex matrix

$$Q = \begin{pmatrix} Q_{+1} & Q_{+2} \\ Q_{-1} & Q_{-2} \end{pmatrix} \equiv \begin{pmatrix} \vec{Q}_+ \\ \vec{Q}_- \end{pmatrix} . \tag{8.56}$$

The holomorphic invariants of the $SO(2)$ part are on the form

$$\begin{aligned} M_{SO(2)} &= \left\{Q^{\mathrm{T}} J Q,\ \det Q\right\} \\ &= \left\{\begin{pmatrix} 2Q_{-1}Q_{+1} & Q_{+1}Q_{-2}+Q_{+2}Q_{-1} \\ Q_{+1}Q_{-2}+Q_{+2}Q_{-1} & 2Q_{+2}Q_{-2} \end{pmatrix},\ Q_{+1}Q_{-2}-Q_{+2}Q_{-1}\right\}. \end{aligned} \tag{8.57}$$

We have to remove the points $\vec{Q}_+ = 0$ and $\vec{Q}_- = 0$, where all the holomorphic invariants vanish $M = 0$. The moduli spaces of vacua turn out to be

$$\mathcal{M}^{SO(2)}_{N_{\mathrm{F}}=2} = W\mathbb{C}P^3_{(1,1,-1,-1)} - \{M_{SO(2)} = 0\} = ((\mathbb{C}^2)^*_+ \times (\mathbb{C}^2)^*_-)/\mathbb{C}^*\,, \tag{8.58}$$

$$\mathcal{M}^{U(1)\times SO(2)}_{N_{\mathrm{F}}=2} = \left((\mathbb{C}^2)^*/\mathbb{C}^*\right) \times \left((\mathbb{C}^2)^*/\mathbb{C}^*\right) = \mathbb{C}P^1 \times \mathbb{C}P^1. \tag{8.59}$$

Since positive real eigenvalues λ_1 and λ_2 satisfy $\sqrt{\lambda_1} + \sqrt{\lambda_2} = \sqrt{\lambda_1 + \lambda_2 + 2\sqrt{\lambda_1\lambda_2}}$, the Kähler potential can be easily shown to be

$$K^{SO(2)}_{N_{\mathrm{F}}=2} = \sqrt{\mathrm{Tr}MM^\dagger + 2\sqrt{\det MM^\dagger}} = 2\sqrt{|\vec{Q}_+|^2|\vec{Q}_-|^2}\,, \tag{8.60}$$

$$K^{U(1)\times SO(2)}_{N_{\mathrm{F}}=2} = \frac{\xi}{2}\log|\vec{Q}_+|^2 + \frac{\xi}{2}\log|\vec{Q}_-|^2\,. \tag{8.61}$$

The prefactor $\xi/2$ in Eq. (8.61) will turn out to have a significant difference from the usual prefactor ξ of the Kähler potential for usual $\mathbb{C}P^1$, see Eq. (1.113), when we will consider 1/2 BPS solitons.

It is straightforward to extend this to the case with generic N_{F}. The manifolds are on the form

$$\begin{aligned} \mathcal{M}^{SO(2)}_{N_{\mathrm{F}}} &= W\mathbb{C}P^{2N_{\mathrm{F}}-1}_{(1_{N_{\mathrm{F}}},-1_{N_{\mathrm{F}}})} - \{M_{SO(2)} = 0\} = ((\mathbb{C}^{N_{\mathrm{F}}})^*_+ \times (\mathbb{C}^{N_{\mathrm{F}}})^*_-)/\mathbb{C}^*\,, \\ \mathcal{M}^{U(1)\times SO(2)}_{N_{\mathrm{F}}} &= \left((\mathbb{C}^{N_{\mathrm{F}}})^*/\mathbb{C}^*\right) \times \left((\mathbb{C}^{N_{\mathrm{F}}})^*/\mathbb{C}^*\right) = \mathbb{C}P^{N_{\mathrm{F}}-1} \times \mathbb{C}P^{N_{\mathrm{F}}-1}\,. \end{aligned} \tag{8.62}$$

The Kähler potential for the latter manifold can be obtained by merely replacing the two vectors $Q_{1,2}$ by N_{F} vectors in Eq. (8.61). Then the meson field becomes an N_{F}-by-N_{F} matrix, however, only two eigenvalues λ_1, λ_2 of $MM^\dagger$ take non-zero values and in this case we have the following identity

$$\det(\lambda \mathbf{1}_{N_{\mathrm{F}}} - MM^\dagger) = \lambda^{N_{\mathrm{F}}-2}\det\left(\lambda\mathbf{1}_2 - (QQ^\dagger)J^\dagger(QQ^\dagger)^{\mathrm{T}}J\right)\,. \tag{8.63}$$

From this characteristic polynomial, we can read off

$$\lambda_1 + \lambda_2 = 2|\vec{Q}_+|^2|\vec{Q}_-|^2 + 2|\vec{Q}_+\vec{Q}_-^\dagger|^2\,, \tag{8.64}$$

$$\lambda_1\lambda_2 = \left(|\vec{Q}_+|^2|\vec{Q}_-|^2 - |\vec{Q}_+\vec{Q}_-^\dagger|^2\right)^2\,. \tag{8.65}$$

Therefore, we find also in the case of N_{F} flavors

$$K^{SO(2)}_{N_{\mathrm{F}}} = \sqrt{\lambda_1} + \sqrt{\lambda_2} = 2\sqrt{|\vec{Q}_+|^2|\vec{Q}_-|^2}\,. \tag{8.66}$$

8.4.2 The $USp(2)$ quotient

This case completely reduces to the $SU(2)$ case with $N_{\rm F}$ flavors. It is not difficult to show that only two eigenvalues λ_1, λ_2 of $MM^\dagger$ take non-zero values and they coincide

$$\lambda_1 = \lambda_2 = \frac{1}{2}{\rm Tr}[MM^\dagger] = \det(QQ^\dagger) \,, \tag{8.67}$$

and this indeed yields the Kähler potential for the $SU(2)$ case

$$K_{N_{\rm F}}^{USp(2)\simeq SU(2)} = {\rm Tr}[\sqrt{MM^\dagger}] = 2\sqrt{\det(QQ^\dagger)} \,. \tag{8.68}$$

We find explicitly the $\mathbb{Z}_2$-conifold singularity at the origin in this model.

8.4.3 The $USp(4)$ quotient

By "diagonalizing" M by $M_{ij} = \mu_i J_{ij}$, we find two non-vanishing eigenvalues both with multiplicity two, that is $\lambda_1 = \lambda_3 = \mu_1^2$ and $\lambda_2 = \lambda_4 = \mu_2^2$ and they can be written as

$$\lambda_1 + \lambda_2 = \frac{1}{2}{\rm Tr}[MM^\dagger] \,, \quad \lambda_1\lambda_2 = \sum_{\langle A\rangle} |P_{\langle A\rangle}|^2 \,, \tag{8.69}$$

where $P_{\langle A\rangle}$ is the Pfaffian of a minor matrix

$$P_{\langle A_1A_2A_3A_4\rangle} \equiv 3M_{A_1[A_2}M_{A_3A_4]} \,. \tag{8.70}$$

In this case where we have $USp(4)$ i.e. $M = 2$, thus it can be written as

$$\sum_{\langle A\rangle} |P_{\langle A\rangle}|^2 = \frac{1}{8}\left({\rm Tr}[MM^\dagger]\right)^2 - \frac{1}{4}{\rm Tr}[(MM^\dagger)^2] \,. \tag{8.71}$$

Since the right hand sides of both the equations in Eq. (8.69) are invariant under the flavor transformation performing the diagonalization, we find for generic number of flavors $N_{\rm F}$

$$K_{N_{\rm F}}^{USp(4)} = 2\left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right) = 2\sqrt{\frac{1}{2}{\rm Tr}[MM^\dagger] + 2\sqrt{\sum_{\langle A\rangle}|P_{\langle A\rangle}|^2}} \,. \tag{8.72}$$

Considering a minimal case with $M_{\rm F} = M = 2$, with the following parametrization

$$M = \begin{pmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ -\phi_1 & 0 & \chi_3 & -\chi_2 \\ -\phi_2 & -\chi_3 & 0 & \chi_1 \\ -\phi_3 & \chi_2 & -\chi_1 & 0 \end{pmatrix} \,, \tag{8.73}$$

we find ${\rm Pf}(M) = \vec{\phi}\cdot\vec{\chi}$ and the simple form of the Kähler potential

$$K_{N_{\rm F}=4}^{USp(4)} = 2\sqrt{\frac{1}{2}{\rm Tr}[MM^\dagger] + 2|{\rm Pf}(M)|} = 2\sqrt{|\vec{\phi}|^2 + |\vec{\chi}|^2 + 2|\vec{\phi}\cdot\vec{\chi}|} \,. \tag{8.74}$$

Manifestly, we can observe an orbifold singularity on the submanifold

$$|\vec{\phi}|^2 + |\vec{\chi}|^2 \neq 0 \,, \quad \mathrm{Pf}(M) = \vec{\phi}\cdot\vec{\chi} = 0 \,, \tag{8.75}$$

of which the rank is $2M-2=2$, since the $\mathrm{Pf}(M) \in \mathbb{C}$ is an appropriate coordinate describing the orthogonal direction to the submanifold and the term $\sqrt{|\mathrm{Pf}(M)|^2}$ emerges in the potential. In a generic region away from this singular submanifold, the scalar curvature is given by

$$R = \frac{20}{\sqrt{|\vec{\phi}|^2 + |\vec{\chi}|^2 + 2|\vec{\phi}\cdot\vec{\chi}|}} \,, \tag{8.76}$$

and is finite even in the vicinity of the submanifold.

8.4.4 The $SO(3)$ quotient

The Kähler quotient for $SO(3)$ with N_{F} flavors reads

$$K^{SO(3)}_{N_{\mathrm{F}}} = \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3} \,, \tag{8.77}$$

and it is obtained by solving the following algebraic equations

$$(K^2 - A_1)^2 = 4A_2 + 8\sqrt{A_3}K \,, \tag{8.78}$$

where the definitions are

$$\begin{aligned}
A_1 &\equiv \lambda_1+\lambda_2+\lambda_3 = \mathrm{Tr}[MM^\dagger] \,, \\
A_2 &\equiv \lambda_1\lambda_2+\lambda_3\lambda_2+\lambda_3\lambda_1 = \frac{1}{2}(\mathrm{Tr}[MM^\dagger])^2 - \frac{1}{2}\mathrm{Tr}[(MM^\dagger)^2] \,, \\
A_3 &\equiv \lambda_1\lambda_2\lambda_3 \,.
\end{aligned} \tag{8.79}$$

A solution with a real number satisfying $K^2 \geq A_1 > 0$ should be unique. Here $\sqrt{A_3}$ does not imply a singularity immediately. In the case of $N_{\mathrm{F}} = N = 3$, we can rewrite it in terms of the baryon field B as

$$\sqrt{A_3} = \sqrt{\det(MM^\dagger)} = \sqrt{|\det M|^2} = |B|^2 \,, \tag{8.80}$$

and around the submanifold with $B=0$, B is an appropriate coordinate around the submanifold. With $K_0 = K|_{|B|^2=0}$, we find

$$K^{SO(3)}_{N_{\mathrm{F}}=3} = K_0 + \frac{2|B|^2}{K_0^2 - A_1} + \mathcal{O}(|B|^4) \,. \tag{8.81}$$

Since $K_0^2 - A_1 = 0$ implies that $A_2 = |B|^2 = 0$, which in turn implies that $\mathrm{rank}\, M \geq N-2 = 1$, this expansion tells us that the submanifold with $\mathrm{rank}\, M = N-1 = 2$ is not singular.

Let us now consider this simple example of $SO(3)$ with $N_{\mathrm{F}} = 2$. The result of the Kähler potential is the same as in the $SO(2)$ case with $N_{\mathrm{F}} = 2$

$$K^{SO(3)}_{N_{\mathrm{F}}=2} = \sqrt{\mathrm{Tr}\, MM^\dagger + 2|\det M|} \,. \tag{8.82}$$

8.5 The $SO(N)$ and $USp(2M)$ hyper-Kähler quotients

Our next task is lifting up the $SO(N)$ and $USp(N = 2M)$ Kähler quotients of the previous Subsection to the hyper-Kähler quotients as we did for the $U(N)$ (hyper-)Kähler quotient in Sec. 1.3. We leave the issues of the hyper-Kähler quotients of $U(1)\times SO(N)$ and $U(1) \times USp(2M)$ for the end of this Chapter. In order to construct the $SO(N)$, $USp(2M)$ hyper-Kähler quotient we need to consider $\mathcal{N} = 2$ hypermultiplets. Hence, we consider an $\mathcal{N} = 2$ extension of the $\mathcal{N} = 1$ Kähler potential (8.16), together with the superpotential

$$\tilde{K}_{SO,USp} = \mathrm{Tr}\left[QQ^\dagger e^{-V'} + \tilde{Q}^\dagger\tilde{Q}e^{V'} + \lambda\left(e^{-V'^{\mathrm{T}}}Je^{-V'} - J\right)\right] , \tag{8.83}$$

$$W = \mathrm{Tr}\left[Q\tilde{Q}\Sigma' + \chi\left(\Sigma'^{\mathrm{T}}J + J\Sigma'\right)\right] , \tag{8.84}$$

where (V', Σ') denote the $SO(N)$ or $USp(2M)$ vector multiplets, $(Q, \tilde{Q}^\dagger)$ are N_{F} hypermultiplets in the fundamental representation of $SO(N)$ or $USp(2M)$, and (λ, χ) are the Lagrange multipliers which are N-by-N matrix valued superfields.

We can rewrite the Kähler potential (8.83) as follows

$$\begin{aligned}\tilde{K}_{SO,USp} &= \mathrm{Tr}\left[QQ^\dagger e^{-V'} + J^{\mathrm{T}}e^{-V'}J\tilde{Q}^{\mathrm{T}}\tilde{Q}^*\right]\\ &= \mathrm{Tr}\left[\mathcal{Q}\mathcal{Q}^\dagger e^{-V'}\right] , \quad \mathcal{Q} \equiv \left(Q,\ J\tilde{Q}^{\mathrm{T}}\right) ,\end{aligned} \tag{8.85}$$

where we have used $e^{V'^{\mathrm{T}}} = J^{\mathrm{T}}e^{-V'}J$. This Kähler potential is nothing but the $\mathcal{N} = 1$ Kähler potential of $SO(N)$ and $USp(2M)$ with $\mathcal{Q}$, a set of $2N_{\mathrm{F}}$ chiral superfields. We can straightforwardly borrow the result of Sec. 8.2 and hence the Kähler potential reads

$$\tilde{K}_{SO,USp} = \mathrm{Tr}\left[\sqrt{\mathcal{M}\mathcal{M}^\dagger}\right] , \qquad \mathcal{M} \equiv \mathcal{Q}^{\mathrm{T}}J\mathcal{Q} . \tag{8.86}$$

The constraint coming from the superpotential (8.84) is

$$Q\tilde{Q}J = J\tilde{Q}^{\mathrm{T}}Q^{\mathrm{T}} \quad \Rightarrow \quad \mathcal{Q}\tilde{J}\mathcal{Q}^{\mathrm{T}} = 0 , \quad \text{with } \tilde{J} \equiv \begin{pmatrix} \mathbf{0} & \mathbf{1}_{N_{\mathrm{F}}} \\ -\epsilon\mathbf{1}_{N_{\mathrm{F}}} & \mathbf{0}\end{pmatrix} . \tag{8.87}$$

Therefore, we again find the constraints for the meson field $\mathcal{M}$

$$\mathcal{M}^{\mathrm{T}} = \epsilon\mathcal{M} , \quad \mathcal{M}\tilde{J}\mathcal{M} = 0 , \quad N-2 < \mathrm{rank}\,\mathcal{M} \le N . \tag{8.88}$$

As well-known, the $SO(N)$ case has a $USp(2N_{\mathrm{F}})$ flavor symmetry while the $USp(2M)$ case has an $O(2N_{\mathrm{F}})$ flavor symmetry. Therefore the $USp(2N_{\mathrm{F}})$ and $O(2N_{\mathrm{F}})$ isometries act on the $SO(N)$ and $USp(2M)$ hyper-Kähler quotients, respectively. The resultant spaces can be written locally in generic points as

$$\mathcal{M}^{\mathrm{HK}}_{SO(N)} \simeq \mathbb{R}^N_{>0}\times\frac{USp(2N_{\mathrm{F}})}{USp(2N_{\mathrm{F}}-2N)\times(\mathbb{Z}_2)^{N-1}} \supset \mathbb{R}^N_{>0}\times\frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}}-N)\times(\mathbb{Z}_2)^{N-1}} , \tag{8.89}$$

$$\mathcal{M}^{\mathrm{HK}}_{USp(2M)} \simeq \mathbb{R}^M_{>0}\times\frac{SO(2N_{\mathrm{F}})}{SO(2N_{\mathrm{F}}-4M)\times USp(2)^M} \supset \mathbb{R}^M_{>0}\times\frac{U(N_{\mathrm{F}})}{U(N_{\mathrm{F}}-2M)\times USp(2)^M} , \tag{8.90}$$

for the $SO(N)$ and $USp(2M)$ hyper-Kähler quotients, respectively. These are hyper-Kähler spaces of cohomogeneity N and M, respectively.[6] The right-most ones denote the corresponding $SO(N)$ and $USp(2M)$ Kähler quotients given in Eqs. (8.5) and (8.10), respectively. These Kähler spaces are special Lagrangian subspaces of the hyper-Kähler spaces. As in the Kähler cases (8.5) and (8.10), the isotropy (unbroken flavor symmetry) changes from point to point. It is enhanced when some eigenvalues coincide.

Let us make a comment on the relation to the instanton moduli space. In Eq. (8.90) the simplest case of the $USp(2) \simeq SU(2)$ hyper-Kähler quotient was previously found in Ref. [58] to be

$$\mathcal{M}^{\mathrm{HK}}_{USp(2)\simeq SU(2)} \simeq \mathbb{R}_{>0} \times \frac{SO(2N_{\mathrm{F}})}{SO(2N_{\mathrm{F}}-4) \times USp(2)} \,. \tag{8.91}$$

This is a hyper-Kähler cone and is particularly important because the single instanton moduli space of an $SO(2N_{\mathrm{F}})$ gauge theory is the direct product of this space and $\mathbb{C}^2$ i.e. the position. Here $\mathbb{R}_{>0}$ parametrizes the size while the coset part parametrizes the orientation of a single BPST instanton embedded into the $SO(2N_{\mathrm{F}})$ gauge group. The moduli space of k instantons in $SO(N)$ and $USp(2M)$ gauge theories are known to be given by $USp(2k)$ and $O(k)$ hyper-Kähler quotients, respectively [24, 60, 61]. Compared with our spaces in Eqs. (8.89) and (8.90), the instanton moduli spaces contain adjoint fields of $USp(2k)$ and $O(k)$ too and thus are larger. Inclusion of adjoint fields remains as a difficult but important problem.

8.6 Discussion

We have explicitly constructed the Kähler potentials for NLσMs describing the Higgs phase of $\mathcal{N}=1$ supersymmetric $SO(N)$ and $USp(2M)$ gauge theories. The key point in the construction lies in the use of taking the gauge symmetry to be $U(N)$ and restricting the algebra down to either $\mathfrak{so}(N)$ or $\mathfrak{usp}(2M)$ with Lagrange multipliers. The result is written both in terms of the component fields and the holomorphic invariants, i.e. the mesons and the baryons of the theories. Because the obtained result is difficult to manage in practice in the large N (N_{F}) limit, we have developed an expansion around the vacuum expectation values of the meson field, and obtained the scalar curvature of both theories, i.e. $SO(N)$ and $USp(2M)$. Furthermore, we have made the same considerations for the case of $U(1) \times SO(N)$ and $U(1) \times USp(2M)$, and obtained the Kähler potential, metric, expansion and curvature also in these cases.

Following the same strategy as in the Kähler quotient case, we have been able to obtain the hyper-Kähler quotient in the case of $SO(N)$ and $USp(2M)$ gauge theories, simply by rewriting the fields by means of the algebra, to fields with $2N_{\mathrm{F}}$ flavors, all in the fundamental representation and we confirm the flavor symmetry of the $SO(N)$ hyper-Kähler quotient to be $USp(2N_{\mathrm{F}})$ and for $USp(2M)$ it is $O(2N_{\mathrm{F}})$.

[6] Any smooth hyper-Kähler manifold of cohomogeneity one, must be the cotangent bundle over the projective space, $T^\star\mathbb{C}P^{N_{\mathrm{F}}-1}$ or flat space [244]. For the $U(1)$ hyper-Kähler quotient with N_{F} flavors, the space is of cohomogeneity one: $\mathbb{R}_{>0} \times SU(N_{\mathrm{F}})/SU(N_{\mathrm{F}}-2)$. This space is blown up to a smooth manifold $T^\star\mathbb{C}P^{N_{\mathrm{F}}-1}$ once the FI parameters are introduced for the $U(1)$ gauge group. The result of Ref. [244] implies that hyper-Kähler spaces of cohomogeneity one in Eqs. (8.89) and (8.90) must have a singularity.

A significant feature of those NLσMs, is that a point in the target space can reach within a finite distance, submanifolds corresponding to unbroken phases of the gauge theories. We have observed that a curvature singularity emerges there. If we consider a generic gauge group with a generic representation as the original gauge theory, we can observe such singularities in many NLσMs unlike the well-known $U(N)$ (Grassmannian) case. The NLσMs we have considered here can be regarded as test cases for those theories.

Before closing this Chapter we would like to comment on the hyper-Kähler quotient of $U(1) \times SO(N)$ and $U(1) \times USp(2M)$. We succeeded in constructing the hyper-Kähler quotient of $SO(N)$ and $USp(2M)$ thanks to the fact that $J\tilde{Q}^{\mathrm{T}}$ is in the fundamental representation, which is the same representation as Q. Although, we want to make use of the same strategy for $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ as before, $J\tilde{Q}^{\mathrm{T}}$ still has charge -1 with respect to the $U(1)$ gauge symmetry while Q has $U(1)$ charge $+1$. Therefore, it is not easy to construct the $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ hyper-Kähler quotient and we will not solve this problem here.

An extension to hyper-Kähler quotients with other gauge groups, namely exceptional groups is also an interesting future problem. As in Eq. (8.16) for $SO(N)$ and $USp(2M)$ Kähler quotients, they may be achieved by introducing a proper constraint. For instance for an E_6 quotient, $\Gamma_{ijk}\big(e^{V'}\big)^i{}_l\big(e^{V'}\big)^j{}_m\big(e^{V'}\big)^k{}_n - \Gamma_{lmn} = 0$ is a candidate constraint to embed E_6 into $U(27)$, where Γ_{ijk} is the third-rank invariant symmetric tensor of E_6. This will be achieved by introducing a Lagrange multiplier λ^{lmn} belonging to the rank-3 anti-symmetric representation. Since the study of vortices in $U(1) \times G'$ with G' being exceptional groups has been raised in Ref. [6], lumps in these Kähler quotients are also interesting subjects to be studied.

We should also consider hyper-Kähler quotients for other representations. In particular, including adjoint fields into our work is important because the resultant spaces appear as multi-instanton moduli spaces of $SO(N)$ and $USp(2M)$ gauge theories.

In certain models it has been proposed that the moduli space of vacua admits a Ricci-flat (non-compact Calabi-Yau) metric [236]. In the case of the $SU(N_{\mathrm{C}})$ Kähler quotient, a Ricci-flat metric was obtained by deforming the Kähler potential (1.95) of the original $SU(N_{\mathrm{C}})$ gauge theory to $K = f\left(\mathrm{Tr}[QQ^\dagger e^{-V'}]\right)$ with an unknown function f, and solving the Ricci-flat condition (the Monge-Ampère equation) for f [218]. The metric turns out to be the canonical line bundle over the Grassmann manifold $Gr_{N_{\mathrm{F}},N_{\mathrm{C}}}$ [245]. It is certainly worthwhile to construct a Ricci-flat metric also on the SO and USp Kähler quotients.

In the case of the $\mathcal{N} = 2$ hyper-Kähler NLσM, the only possible potential consistent with eight supercharges is written as the square of a tri-holomorphic Killing vector [246, 247]. The explicit potentials can be found for instance for $T^\star\mathbb{C}P^{N-1}$ [248, 249, 250], toric hyper-Kähler manifolds [251], $T^\star Gr_{N,M}$ [167, 168] and $T^\star F_n$ [70]. In terms of the hyper-Kähler quotients these potentials are obtained as usual masses of hypermultiplets in the corresponding $\mathcal{N} = 2$ supersymmetric gauge theories [167, 168]. For this massive deformed hyper-Kähler NLσM one can construct domain walls which are other fundamental 1/2 BPS objects; 1/2 BPS domain wall solutions in the $U(N)$ hyper-Kähler quotient, namely $T^\star Gr_{N,N}$, see Refs. [29, 98, 99]. Constructing a massive deformation and domain wall solutions in $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ hyper-Kähler quotients remain as future problems.

CHAPTER 9

Non-linear σ model lumps

The 1/2 BPS lumps in the $U(1) \times SO(2M)$ and $U(1) \times USp(2M)$ Kähler quotients and their effective descriptions are studied in this Chapter. In this connection, a general relation between the moduli spaces of vortices and lumps is discussed. We find a new singular limit of the lumps with non-vanishing sizes in addition to the ordinary small lump singularity. The former is due to the existence of singular submanifolds in the target spaces. This fact is also important with respect to fractional vortices and lumps. Finally, we identify the normalizable zero-modes of a single lump configuration (with lump number one).

9.1 Lumps in $U(1) \times G'$ Kähler quotients

In this Chapter we will study non-linear σ model (NLσM) lumps which are 1/2 BPS configurations. Lumps are stringy topological textures extending for instance in the x^3 direction in $d = 1 + 3$ dimensional spacetime and are supported by the non-trivial second homotopy group $\pi_2(\mathcal{M})$ associated with a holomorphic map from the 2 dimensional spatial plane $z = x_1 + ix_2$ to a 2-cycle of the target space of the NLσM. We will consider the $\mathbb{C}$-plane together with the point at infinity, that is $z \in \mathbb{C} \cup \{\infty\} \simeq S^2$, which is mapped into the target space. Lumps in non-supersymmetric $SO(N)$ theories were studied in Refs. [228, 229] where the second homotopy group is $\pi_2[SU(N)/SO(N)] \simeq \mathbb{Z}_2$ and therefore those lumps are non-BPS. Here we do not consider this type of lumps. We will first study BPS lumps in the NLσM of $U(1) \times G'$ Kähler quotients in general and then we investigate lumps in the case of $G' = SO, USp$ which have been constructed in Chap. 8.

In the NLσM of $U(1) \times G'$ Kähler quotients, (inhomogeneous) complex coordinates $\{\phi^\alpha\}$ of the Kähler manifold, which are the lowest scalar components of the chiral superfields, are given by some set of holomorphic G' invariants I^i modulo $U(1)^{\mathbb{C}}$, namely $\phi^\alpha \in \{I^i\} /\!\!/ U(1)^{\mathbb{C}}$. Static lump solutions can be obtained by just imposing ϕ^α to be a holomorphic function with respect to z

$$\phi^\alpha(t, z, \bar{z}, x^3) \to \phi^\alpha(z; \varphi^i) \ , \tag{9.1}$$

where φ^i denote complex constants. The tension of the lumps can be obtained by plugging the solution back into the Lagrangian

$$T = 2 \int_{\mathbb{C}} K_{\alpha\bar{\beta}}(\phi, \bar{\phi}) \, \partial\phi^\alpha \bar{\partial}\bar{\phi}^{\bar{\beta}} \Big|_{\phi \to \phi(z)} = 2 \int_{\mathbb{C}} \bar{\partial}\partial K(\phi, \bar{\phi}) \Big|_{\phi \to \phi(z)} \ , \tag{9.2}$$

where K is the Kähler potential and $K_{\alpha\bar{\beta}} = \partial_\alpha \bar{\partial}_{\bar{\beta}} K$ is the Kähler metric. We would like to stress that all the parameters φ^i are nothing but the moduli parameters of the 1/2 BPS lumps.

We assume that the boundary of $z \to \infty$ is mapped to a single point $\phi^\alpha(z) \to \phi^\alpha_{\rm vev}$ on the target space. Since the functions $\phi^\alpha(z)$ should be single valued, $\phi^\alpha(z)$ can be expressed with a finite number of poles as

$$\phi^\alpha(z) = \phi^\alpha_{\rm vev} + \sum_{i=1}^{k} \frac{\phi^\alpha_i}{z - z_i} + \mathcal{O}(z^{-2}) \ . \tag{9.3}$$

Strictly speaking, we have to change patch of the target manifold at the poles to describe the solutions correctly. To describe the lump solutions, it is convenient to use the holomorphic G' invariants I^i satisfying the constraints as homogeneous coordinates. The holomorphic map is expressed by the homogeneous coordinates $I^i(z)$ which are holomorphic in z

$$I^i(z) = I^i_{\rm vev} z^{n_i \nu} + \mathcal{O}(z^{n_i \nu - 1}) \ , \tag{9.4}$$

where n_i is the $U(1)$ charge of the holomorphic G' invariant I^i, and ν is some number. $I^i_{\rm vev}$ denotes the vacuum expectation value of I^i at spatial infinity. Since all $n_i\,\nu$ must take value in $\mathbb{Z}_{>0}$, we can express $\nu = k/n_0$ with the greatest common divisor (GCD) n_0 of $\{n_i\}$ and k a non-negative integer. The integer k will be found to be the topological winding number. These polynomials are the basic tools to study lump solutions and their moduli and $\phi^\alpha(z)$ can be written as ratios of these polynomials, namely $U(1)^{\mathbb{C}}$ invariants, which are known as rational maps in the Abelian case.

There is a remark in store for constructing lump solutions. If a holomorphic map (9.4) touches the unbroken phase of the original gauge theory at some point, the behavior of the lump is ill-defined there in terms of the NLσM. Generally speaking, as we will see in examples later, the lump configuration becomes singular at that point. Therefore, we have to exclude such singular configurations and *all points in the base manifold* $\mathbb{C}$ *must be mapped to the full Higgs phase by the holomorphic map (9.4).* We will denote this condition the *lump condition*. In other words, there exist limits where lump configurations become singular by varying the moduli parameters. For instance, the invariants $I^i(z)$ are prohibited from having common zeros by the lump condition. Since common zeros cannot be detected even in the vicinity of a corresponding point in the base space, an emergence of common zeros indicates a small lump singularity, which is well-known for lumps in the $\mathbb{C}P^n$ model. The lump condition requires non-vanishing size moduli there. As we will show in examples later, this situation implies the emergence of a local vortex. The lump condition is stronger than the condition of no common zeros in the invariants, except for the $U(N)$ case [116], where in fact both the conditions are equivalent. The difference between the two conditions above implies the existence of limits where a lump configuration becomes singular with a non-vanishing size. This is a typical property of lumps in a NLσM with a singular submanifold. We will see explicit examples of this property later.

9.2 Lump moduli spaces vs. vortex moduli spaces

As a NLσM can be obtained in the strong gauge coupling limit of the gauge theory, lump solutions in such NLσMs can also be given as that limit of semi-local vortex solutions, whose configurations can smoothly be mapped to the Higgs phase. Therefore, lump solutions are closely related to semi-local vortices in the original gauge theory, even with a finite gauge coupling. Lumps in the $U(N)$ Kähler quotient, namely in the Grassmann σ model, have been studied previously in Refs. [75, 76, 77, 115, 116]. In fact, the dimensions of both the moduli spaces coincide $\mathrm{dim}_{\mathbb{C}}\mathcal{M}^{k\text{-vortex}}_{U(N),N_{\rm F}} = \mathrm{dim}_{\mathbb{C}}\mathcal{M}^{k\text{-lump}}_{U(N),N_{\rm F}} = kN_{\rm F}$ [9, 136, 252]. It has been found that the moduli space of k lumps in the Grassmann σ model is identical to that of k semi-local vortices with the lump condition in Ref. [116]. Hence, the inclusive relation is $\mathcal{M}^{k\text{-vortex}}_{U(N),N_{\rm F}} \supset \mathcal{M}^{k\text{-lump}}_{U(N),N_{\rm F}}$. The lump condition excludes subspaces of $\mathcal{M}^{k\text{-vortex}}_{U(N),N_{\rm F}}$ corresponding to the minimal size vortices whose size is of order of the inverse gauge coupling.

In this Section we will discuss the relation between moduli spaces for lump solutions and vortex solutions in the $U(1)\times SO(N)$ and $U(1)\times USp(2M)$ cases. Here we take $N = N_{\rm F}$ and $\det M_{\rm vev} \neq 0$ for simplicity. The dimension of the moduli space of k vortices in a $U(1)\times G'$ gauge theory ($N_{\rm F} = N$) has been found to be [6]

$$\mathrm{dim}_{\mathbb{C}}\mathcal{M}^{k\text{-vortex}}_{U(1)\times G'} = \frac{kN^2}{n_0}\ , \tag{9.5}$$

with $N = 2M$ for $USp(2M)$. In the following, we will count the dimensions of the lump moduli spaces. (We will use the same characters for the lowest scalar components of chiral superfields as for the superfields themselves).

In the $U(1)\times SO(2M)$ case $(N = 2M)$, lump solutions with fixed boundary conditions are given by taking the following polynomials as the holomorphic invariants $I^i = \{M, B\}$ defined in (8.22). Their $U(1)$ charges are $\{2, 2M\}$, respectively. Thus, their GCD is $n_0 = 2$ and we find

$$M(z) = M_{\rm vev}z^k + \mathcal{O}(z^{k-1})\ , \quad B(z) = B_{\rm vev}z^{kM} + \mathcal{O}(z^{kM-1})\ , \tag{9.6}$$

with $k \in \mathbb{Z}_{>0}$. Note that we should not neglect the baryon field B, although the baryon field B is dependent on M. This is because the baryon field B determined by $M(z)$ is not necessarily holomorphic everywhere in the complex plane $\mathbb{C}$:

$$\det(J)B(z)^2 = \det M(z)\ . \tag{9.7}$$

Generically, this gives $2kM$ constraints for moduli parameters. For instance, with a single lump solution in the $U(1)\times SO(2)$ case, a general form of $M(z)$ is given by setting $M_{\rm vev} = \sigma_1$ and $k = 1$

$$M(z) = \begin{pmatrix} b & z-a \\ z-a & c \end{pmatrix} \quad \rightarrow \quad \det M(z) = bc - (z-a)^2\ . \tag{9.8}$$

The constraint (9.7) requires $\det M(z)$ to be exactly a square of a polynomial and then we find the non-trivial conditions; $b = 0$ or $c = 0$ where the intersection point $b = c = 0$ is excluded by the lump condition. These two disconnected solutions correspond to two different types of lumps wrapping different $\mathbb{C}P^1$'s of $\mathcal{M}^{U(1)\times SO(2)}_{N_{\rm F}=2} = \mathbb{C}P^1\times\mathbb{C}P^1$ in Eq. (8.59). For generic k-lump configurations, we can count the degrees of freedom of the moduli parameters as

$$\begin{aligned} \mathrm{dim}_{\mathbb{C}}\,\mathcal{M}^{k\text{-lump}}_{SO(2M)} &= \#\text{moduli in } M(z) + \#\text{moduli in } B(z) - \#\text{constraints} \\ &= k\frac{(2M)(2M+1)}{2} + kM - 2kM = 2kM^2\ . \end{aligned} \tag{9.9}$$

In the $U(1)\times SO(2M+1)$ case, the $U(1)$ charges of the invariants $\{M, B\}$ are $\{2, 2M+1\}$. Hence their GCD is $n_0 = 1$ and lump solutions are given by the following polynomials

$$M(z) = M_{\text{vev}} z^{2k} + \mathcal{O}(z^{2k-1}) \,, \tag{9.10}$$

$$B(z) = B_{\text{vev}} z^{(2M+1)k} + \mathcal{O}(z^{(2M+1)k-1}) \,. \tag{9.11}$$

The dimension of the k-lump moduli space in this case is generically given by

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{M}^{k\text{-lump}}_{SO(2M+1)} &= 2k\frac{(2M+1)(2M+2)}{2} + k(2M+1) - 2k(2M+1) \\ &= k(2M+1)^2 \,.\end{aligned} \tag{9.12}$$

These two results are the same as those of the 1/2 BPS vortex moduli spaces derived from the index theorem [6], see Eq. (9.5). That is, at least for generic points of the lump moduli space, the moduli for the lump solutions are sufficient to describe the vortex moduli space in the original gauge theory, and there are no internal moduli unlike the orientational moduli $\mathbb{C}P^{N-1}$ of the $U(N)$ case with $N_{\text{F}} = N$ flavors. This property is significantly different from the $U(N)$ case with the minimal number of flavors $N_{\text{F}} = N$, where only local vortices carrying the orientational moduli exist and the strong coupling limit of them are not lumps but singular objects of zero sizes.

In the $U(1) \times USp(2M)$ case, the baryon field is completely described by the meson fields and there are no constraints

$$M(z) = M_{\text{vev}} z^k + \mathcal{O}(z^{k-1}) \,, \quad B(z) = (\text{Pf}J)^{-1}\text{Pf}(M(z)) \,. \tag{9.13}$$

Therefore, the number of complex parameters in $M(z)$ is simply given by

$$\#\text{moduli in } M(z) = k\frac{2M(2M-1)}{2} = \dim_{\mathbb{C}} \mathcal{M}^{k\text{-vortex}}_{USp(2M)} - kM \,. \tag{9.14}$$

Note that it is different from the dimension of the vortex moduli space. This deficit number M for each lump can be understood as follows. In this case, color-flavor symmetries $USp(2)^M \simeq SU(2)^M$ survive even at a generic point in the vacuum as we explained below Eq. (8.9). These surviving symmetries are broken in a vortex configuration and this means that the vortex configuration has orientational moduli $(\mathbb{C}P^1)^M$ as NG modes. These modes are expected to be localized in the Coulomb phase of the original gauge theory, which corresponds to the curvature singularity of the NLσM, and therefore, cannot be detected as moduli of lump solutions in the NLσM. Therefore, roughly speaking, we guess that

$$\mathcal{M}^{k\text{-vortex}}_{USp(2M)} \sim \mathcal{M}^{k\text{-singular lump}}_{USp(2M)} \times (\mathbb{C}P^1)^{kM} \,, \tag{9.15}$$

where $\mathcal{M}^{k\text{-singular lump}}_{USp(2M)}$ is the would-be lump moduli space which is parametrized by the complex parameters in the meson field $M(z)$. Emergence of these internal moduli is strongly related to singular configurations of lumps.[1] Actually, to get regular solutions for lumps in

[1] This situation is similar to the case of a $U(N)$ gauge theory with $N_{\text{F}} = N$ flavors. The gauge theory has a non-Abelian vortex whose internal moduli space is $\mathbb{C}P^{N-1}$. But the strong gauge coupling limit yields a NLσM of only a point and there are no lump solutions.

any NLσM, we have to require the lump condition, which means that the rank of the meson M should be $2M$ everywhere in this $USp(2M)$ case. Therefore, no regular solutions exist in the case of $N_{\rm F} = 2M$, because $\mathrm{Pf}M$ are polynomials in z with order Mk and thus has kM zeros. We will show a concrete example in the next Section. We expect that each of the orientational moduli $\mathbb{C}P^1$ are attached to such zeros and the deficit dimension of $\mathcal{M}^{k\text{-singular lump}}_{USp(2M)}$ should be strongly related to the non-existence of regular solutions. Regular lump solutions require the number of flavors to be greater than $2M$.

In both cases of $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ gauge theories, additional NG zero modes can emerge as the moduli of vortex configurations if we choose special points as the vacuum, $M_{\rm vev}(B_{\rm vev})$. Especially, by choosing $M_{\rm vev} = J$ ($\mu_i = 1$ for all i), the following moduli spaces for a single local vortex were found as [6]

$$\mathcal{M}^{\rm vortex}_{G',\,k=1} \supset \mathcal{M}^{\rm local\ vortex}_{G',\,k=1} = \mathbb{C} \times \frac{G'}{U(M)} \,, \tag{9.16}$$

$(G' = SO(2M), USp(2M))$ which cannot be moduli of single lump configurations.

To completely treat the vortex moduli, including internal moduli, we need to use *the moduli matrix formalism* [136, 252]. This formalism is obtained by merely rewriting the holomorphic gauge invariants $M(z)$, $B(z)$ in terms of the original chiral field $Q(z)$ whose components are also polynomials in the complex coordinate z.[2] The description of the lump solutions with respect to $Q(z)$ is redundant, since $Q(z)$ and $Q'(z)$ determine the same holomorphic maps $M(z)$, $B(z)$, if they are related by a complexified gauge transformation $Q'(z) = V(z)Q(z)$. Therefore we have the following equivalence relation, called the V-equivalence

$$Q(z) \sim V(z)Q(z) \,, \quad V(z) \in U(1)^{\mathbb{C}} \times \left\{SO(N)^{\mathbb{C}}, USp(2M)^{\mathbb{C}}\right\} \,. \tag{9.17}$$

The parameters contained in $Q(z)$ after gauge fixing, parametrize the moduli space of vortices. Conversely, all moduli of vortices including internal moduli are contained in $Q(z)$, and thus $Q(z)$ is denoted *the moduli matrix*. In this formalism the boundary conditions (9.6), (9.11) and (9.13) are interpreted as constraints for the moduli matrix $Q(z)$ [6]

$$\begin{aligned} SO(2M),\, USp(2M): &\quad Q^{\rm T}(z)JQ(z) = M_{\rm vev}z^k + \mathcal{O}(z^{k-1}) \,, \\ SO(2M+1): &\quad Q^{\rm T}(z)JQ(z) = M_{\rm vev}z^{2k} + \mathcal{O}(z^{2k-1}) \,. \end{aligned} \tag{9.18}$$

The constraint (9.7) is of course automatically solved in this formalism. This formalism is apparently independent of the gauge coupling and it is well-defined to require the lump conditions to hold on the vortex moduli space. We expect that a submanifold of the k-vortex moduli space satisfying the lump condition is equivalent to the k-lump moduli space,

$$\mathcal{M}^{k\text{-lump}} \simeq \left\{a | a \in \mathcal{M}^{k\text{-vortex}},\ \text{the lump condition}\right\} . \tag{9.19}$$

[2] The way to derive the moduli matrix here is slightly different from the way used in Ref. [6]. These two ways can be identified by considering BPS vortex solutions in the superfield formulation [113]. The key observation is that the gauge symmetry G in the supersymmetric theory is complexified : $G^{\mathbb{C}}$. Hence, the moduli matrix naturally appears in the superfield formulation, while if we fix $G^{\mathbb{C}}$ in the Wess-Zumino gauge, the scalar field $Q_{\rm wz}$ appears as the usual bosonic component in the Lagrangian. The moduli matrix is usually denoted by the symbol $H_0(z)$ in the literature.

This expectation is quite natural and is enforced by the above observations by counting the dimensions. Because, if we can consider a NLσM as an approximation to the gauge theory with a strong but finite gauge coupling g, a lump solution should describe an approximate configuration of a vortex, whereas a steep configuration with a width of order $1/g\sqrt{\xi}$ is excluded by some UV cutoff $\Lambda < g\sqrt{\xi}$. Of course, to justify this expectation, we need to verify an equivalence[3] between the two formalisms, the moduli matrix formalism and the holomorphic map (9.4) with the constraint on the invariants, under the lump condition. In examples of the next Section, we just assume that this expectation is true. To construct lump solutions for large $N_{\rm F}(N)$, the moduli matrix formalism is somewhat easier than treating $M(z), B(z)$ as they are.

9.3 Lumps in $U(1)\times SO(2M)$ and $U(1)\times USp(2M)$ Kähler quotients

9.3.1 BPS lumps in the $U(1)\times SO(2M)$ Kähler quotient

Let us start with the simplest example in which the gauge group is $U(1)\times SO(2)$ with two flavors $N_{\rm F}=2$. As we have studied in Sec. 8.4.1, the target space is $\mathbb{C}P^1\times\mathbb{C}P^1$. Lump solutions are classified by a pair of integers (k_+, k_-) given as

$$\pi_2\left(\mathcal{M}_{N_{\rm F}=2}^{U(1)\times SO(2)}\right) = \mathbb{Z}\times\mathbb{Z}\ni(k_+,k_-)\,. \tag{9.20}$$

A solution with (k_+, k_-) lumps is given by

$$Q(z) = \begin{pmatrix} Q_1^+(z) & Q_2^+(z) \\ Q_1^-(z) & Q_2^-(z) \end{pmatrix}\,, \tag{9.21}$$

where $Q_{+i}(z), Q_{-i}(z)$ are holomorphic functions of z of degree $k_\pm$, respectively. One can verify that the tension is given by

$$T = 2\int_{\mathbb{C}}\bar\partial\partial K_{U(1)\times SO(2)} = \pi\xi(k_+ + k_-)\equiv\pi\xi k\,, \tag{9.22}$$

where $K_{U(1)\times SO(2)}$ is the Kähler potential given in Eq. (8.61). Interestingly, the tension of the minimal lump $(k_+,k_-)=(1,0),(0,1)$ is half of $2\pi\xi$ which is that of the minimal lump in the usual $\mathbb{C}P^1$ model. A similar observation has been obtained recently in Ref. [6].

Next, we would like to consider lump configurations in slightly more complicated models by considering general $U(1)\times SO(2M)$ Kähler quotients, where we set $M\geq 2$, $N_{\rm F}=2M$ and $M_{\rm vev}=J$. As an example for $k=1$, we take

$$Q_{k=1} = \begin{pmatrix} z\mathbf{1}_M - A & C \\ 0 & \mathbf{1}_M \end{pmatrix}\,, \qquad \left\{\begin{array}{l} A = \mathrm{diag}(z_1, z_2, \cdots, z_M)\,, \\ C = \mathrm{diag}(c_1, c_2, \cdots, c_M)\,. \end{array}\right. \tag{9.23}$$

[3] In the $U(1)\times USp$ and $U(1)\times SO$ cases, we have to verify that the meson field $M(z)$ whose elements are polynomials can be always decomposed in $Q(z)$ whose elements are also polynomials and furthermore that there is no degeneracy of moduli in the construction of $M(z)$ from $Q(z)$ under the lump condition. There is no known proof and it is expected to be technically complicated.

These diagonal choices allow us to treat the invariants as if they were independent invariants of M different $SO(2)$'s. Hence, one can easily find an $SO(2)$ part inside M as

$$\begin{pmatrix} (M)_{i,i} & (M)_{i,i+M} \\ (M)_{i+M,i} & (M)_{i+M,i+M} \end{pmatrix} = \begin{pmatrix} 0 & z-z_i \\ z-z_i & 2c_i \end{pmatrix} , \quad i=1,2,\cdots,M , \tag{9.24}$$

which satisfies the constraint (9.18). Note that non-zero parameters c_i keep the $\mathrm{rank}\, M \geq 2M-1$, even at $z=z_i$. All their eigenvalues are also eigenvalues of $MM^\dagger$

$$\lambda_{i\pm} = |z-z_i|^2 + 2|c_i|^2 \pm 2|c_i|\sqrt{|z-z_i|^2+|c_i|^2} . \tag{9.25}$$

Thus, the Kähler potential in Eq. (8.43) becomes

$$K = \xi \log\left[\sum_{i=1}^{M}\left(\sqrt{\lambda_{i+}}+\sqrt{\lambda_{i-}}\right)\right] = \xi\log\left(2\sum_{i=1}^{M}\sqrt{|z-z_i|^2+|c_i|^2}\right) . \tag{9.26}$$

The energy density is obtained by $\mathcal{E} = 2\partial\bar\partial K$ with this Kähler potential and exhibits an interesting structure. It is proportional to the logarithm of the sum of the square root of $|P_i(z)|^2$, while the known Kähler potential of a $\mathbb{C}P^M$ lump is just the logarithm of the sum of $|P_i(z)|^2$. This difference gives us quite distinct configurations. If we take some c_i to vanish, then we find that the energy density of the configuration becomes singular at $z=z_i$

$$\mathcal{E} = 2\xi\partial\bar\partial\log\left(\sqrt{|z-z_i|^2}+\cdots\right) \sim \text{const.}\times\frac{1}{|z-z_i|} + \mathcal{O}(z^0) . \tag{9.27}$$

This is due to the curvature singularity which appears when the manifold becomes of $\mathrm{rank}\, M = 2M-2$, and in other words, violate the lump condition. Note that this singular configuration has a non-vanishing size, as we mentioned above. If we take all z_i's and all c_i's to be coincident, respectively, we find that the Kähler potential reduces to that of the minimal winding one in the $U(1)\times SO(2)$ model. This suggests that the trace part of C determines the overall size of the configuration and the trace part of A corresponds to the center of mass. As we will explain later, only this trace part of A among the parameters is a normalizable mode in the effective action of the lump.

A single lump in $U(1)\times SO(2M+1)$ might be almost the same as the coincident $k=2$ lumps in $SO(2M)$. However we will not discuss this case in detail.

9.3.2 BPS lumps in the $U(1)\times USp(2M)$ Kähler quotient

Let us first examine a lump solution in the $U(1)\times USp(2)$ theory with $N_{\rm F}=2$. In this case, however, we obtain only local vortices and cannot observe regular lumps in the NLσM since the vacuum is just a point. After fixing the gauge, the chiral field can be expressed as

$$Q(z) = \begin{pmatrix} z-a & 0 \\ b & 1 \end{pmatrix} . \tag{9.28}$$

This matrix yields

$$M = (z-a)\,J , \qquad K = \frac{\xi}{2}\log|z-a|^2 . \tag{9.29}$$

At the center of the vortex, the rank of M always reduces to zero, where the $U(1)$ gauge symmetry is restored. Therefore, solutions are always singular at that point, because we know that $USp(2) \simeq SU(2)$ and the $U(2)$ model with 2 flavors admits only local vortices rather than semi-local vortices which reduce to lumps in the NLσM limit. Indeed, the parameter b which does not appear in M is the orientational modulus of local vortex in the original $U(1) \times USp(2)$ gauge theory and describes $\mathbb{C}P^1$.

As we have mentioned, lump solutions in the case of $M = M_{\rm F}$ always have singular points in the configurations. The simplest non-trivial example for a regular lump is obtained in the case of $U(1) \times USp(4)$ with 6 flavors. A lump (vortex) solution in this case, with the minimal winding ($k = 1$) has $MN_{\rm F} = 12$ complex parameters. Let us consider the following field configuration as a typical minimal example of $k = 1$;

$$Q(z) = \begin{pmatrix} z - z_+ & 0 & 0 & c & a_+ & 0 \\ 0 & z - z_- & -c & 0 & 0 & a_- \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} , \tag{9.30}$$

which gives the following characteristic polynomial

$$\det(\lambda - MM^\dagger) = \lambda^2 \left(\lambda^2 - (R_+^2 + R_-^2 + 4|c|^2)\lambda + R_+^2 R_-^2\right)^2 , \tag{9.31}$$

with $R_\pm = \sqrt{|z - z_\pm|^2 + |a_\pm|^2}$. Then the energy density of the configuration $\mathcal{E}$ is given by

$$\mathcal{E} = 2\partial\bar{\partial} K_{U(1)\times USp(4)}|_{\rm sol} = \xi \partial\bar{\partial} \log\left((R_+ + R_-)^2 + 4|c|^2\right) . \tag{9.32}$$

This configuration is regular everywhere as long as $a_\pm \neq 0$, that is, it satisfies the lump condition. If we choose $a_+ = a_-$ and $z_+ = z_-$, it corresponds to a $\mathbb{C}P^2$ single lump solution.

9.4 Effective action of lumps

Now we have a great advantage thanks to the above superfield formulation of the NLσM. A supersymmetric low energy effective theory on the 1/2 BPS lumps is immediately obtained merely by plugging the 1/2 BPS solution (9.1) into the Kähler potential which we have obtained in the previous Section after promoting the moduli parameters φ to fields on the lump world-volume

$$\phi^\alpha(t, z, \bar{z}, x^3) \to \phi^\alpha(z; \varphi^i(t, x^3)) . \tag{9.33}$$

The resulting (effective) expression for the Kähler potential is

$$\mathcal{K}_{\rm lump} = \int dz d\bar{z}\, K\left(\phi(z, \varphi^i(t, x^3)),\ \phi^\dagger(\bar{z}, \bar{\varphi}^i(t, x^3))\right) . \tag{9.34}$$

Let us make a simple example of the $\mathbb{C}P^1$ σ model which is the strong coupling limit of a $U(1)$ gauge theory with $N_{\rm F} = 2$ flavors $Q = (Q_1,\ Q_2)$. In this case, Q_1 and Q_2 themselves play the role of the holomorphic invariants I^i and the inhomogeneous coordinate is given by $\phi = Q_2/Q_1$. We fix the $U(1)^{\mathbb{C}}$ symmetry in such a way that Q is expressed by

$$Q = (1,\ b)\ . \tag{9.35}$$

From Eq. (1.113), the Kähler potential and the corresponding Lagrangian are of the form

$$K = \xi \log(1+|b|^2)\ , \qquad \mathcal{L} = \xi \frac{|\partial_\mu b|^2}{(1+|b|^2)^2}\ . \tag{9.36}$$

A single 1/2 BPS lump solution in this model is given by

$$Q(z) = (z - z_0,\ a) \quad \leftrightarrow \quad \phi = \frac{a}{z - z_0}\ , \tag{9.37}$$

where z_0 corresponds to the position of the lump and a is its transverse size and phase moduli. To obtain the effective theory of the lump, one needs to promote the moduli matrix as follows

$$Q(z) = (z - z_0,\ a) \quad \rightarrow \quad Q(t,z) = (z - z_0(t),\ a(t))\ . \tag{9.38}$$

Plugging this into the formal expression (9.34), we get the effective theory

$$\begin{aligned}
\mathcal{L}^{\text{eff}} &= \xi \int dz d\bar{z}\ \delta^t \delta_t^\dagger \log\left(|z - z_0(t)|^2 + |a(t)|^2\right) \\
&= \xi \int dz d\bar{z} \left[\frac{|a(t)|^2}{\left(|z - z_0(t)|^2 + |a(t)|^2\right)^2} |\dot{z}_0(t)|^2 + \frac{|z - z_0(t)|^2}{\left(|z - z_0(t)|^2 + |a(t)|^2\right)^2} |\dot{a}(t)|^2 \right] .
\end{aligned} \tag{9.39}$$

The second term in the second line does not converge, thus the size modulus $a(t)$ is not dynamical. Hence, we should fix it by hand as $a(t) = \text{const} \neq 0$. Then the only dynamical field is the translation $z_0(t)$ and the effective action is

$$\mathcal{L}^{\text{eff}}_\infty = \pi \xi |\dot{z}_0(t)|^2\ , \tag{9.40}$$

where $2\pi\xi$ is the tension of the minimal winding solution.

9.5 Identifying non-normalizable modes

We can determine which parameters in $Q(z)$ are localized on lumps and normalizable, and which parameters are non-normalizable. If there exists a divergence in the Kähler potential which cannot be removed by Kähler transformations, it indicates that the moduli parameters included in the divergent terms are non-normalizable. Let us substitute an expansion of the lump solution with respect to z^{-1}

$$\phi^\alpha(z) = \phi^\alpha_{\text{vev}} + \frac{\chi^\alpha}{z} + \mathcal{O}(z^{-2})\ , \quad \chi^\alpha = \sum_{i=1}^{k} \phi^\alpha_i\ , \tag{9.41}$$

into the Kähler potential (9.34) and expand it as well

$$\begin{aligned}\mathcal{K}_{\rm lump} &= \lim_{L\to\infty}\int_{|z|\le L}\Bigg[K(\phi^\alpha_{\rm vev},\bar\phi^{\bar\beta}_{\rm vev}) + \frac{1}{z}\partial_\alpha K\chi^\alpha + \frac{1}{\bar z}\bar\partial_{\bar\alpha}K\bar\chi^{\bar\alpha} + \frac{1}{|z|^2}\partial_\alpha\bar\partial_{\bar\beta}K\chi^\alpha\bar\chi^{\bar\beta} \\ &\qquad\qquad + \mathcal{O}(|z|^{-3})\Bigg] \qquad (9.42)\\ &= \lim_{L\to\infty}\Big[2\pi L^2\, K(\phi_{\rm vev},\bar\phi_{\rm vev}) + 2\pi\log L\;\partial_\alpha\bar\partial_{\bar\beta}K(\phi_{\rm vev},\bar\phi_{\rm vev})\chi^\alpha\bar\chi^{\bar\beta} + \mathcal{O}(1)\Big]\ ,\end{aligned}$$

where L is an infrared cutoff. Thus we can conclude that the moduli parameters included in $\{\phi^\alpha_{\rm vev},\chi^\alpha\}$ are all non-normalizable and the others are normalizable. The modulus a in the last Section is a typical example of χ^α.

For instance, let us take a look at the example (9.24) of the solution for single lumps in the $U(1)\times SO(2M)$ case. The meson field $M(z)$ has the following elements : $(z-z_i)$ and $2c_i$. One can partly construct inhomogeneous coordinates of the manifold in this case by taking ratios from pairs of the elements,

$$\begin{aligned}\phi^i &= \frac{2c_i}{z-z_M} = \frac{2c_i}{z} + \mathcal{O}(z^{-2}), \quad \text{for } 1\le i\le M\ ,\\ \phi^{i+M} &= \frac{z-z_i}{z-z_M} = 1 - \frac{z_i-z_M}{z} + \mathcal{O}(z^{-2}), \quad \text{for } 1\le i\le M-1\ . \qquad (9.43)\end{aligned}$$

Thus the moduli c_i and $z_i - z_M$ are non-normalizable. The only normalizable modulus is $\sum_{i=1}^M z_i/M$ which is the center of mass. This fact is a result of the Kähler metric (8.44) where the trace part of the meson field M does not contribute to the metric. Generally speaking, all moduli of a single lump in the $U(1)\times SO(2M)$ and $U(1)\times USp(2M)$ theories are non-normalizable except for the center of mass and the orientational moduli of local vortex.

9.6 Discussion

In this Chapter we have studied the $1/2$ BPS NLσM lumps in $U(1)\times G'$ gauge theories and observed that we can construct lump solutions straightforwardly if the Kähler potential for the NLσM is given in terms of holomorphic invariants of G'. We found that counting the dimension of these (regular) lump moduli spaces gives the same result as for the semi-local vortex moduli space in the case of $SO(N)$ and $USp(2M)$ theories. This fact enforces our natural expectation that those moduli spaces are homeomorphic to each other except in the subspaces where the lump condition is violated. Furthermore, by considering effective actions within our formalism for the NLσM lumps, we have obtained a conventional method to clarify the non-normalizability of the moduli parameters in general cases. By using this, we can conclude that in both the cases of $U(1)\times SO(2M)$ and $U(1)\times USp(2M)$ Kähler quotients, all moduli parameters of a single regular lump are non-normalizable except for the center of mass.

An important observation concerning lump configurations in $U(1)\times SO(N)$ and $U(1)$ $\times$ $USp(2M)$ theories is the existence of a singularity in the target manifold. In those theories, a lump configuration becomes singular without taking the zero size limit, simply if the configuration touches the singularity of the manifold, whereas a lump in the $U(N)$ case is always regular with a finite size and becomes singular only in the zero size limit. Especially, in the case of $U(1) \times USp(2M)$ with $N_{\rm F} = 2M$, only singular solutions (with a finite or zero size) exist.

It is an important problem to determine the second homotopy group $\pi_2(\mathcal{M}_{U(1)\times(SO,USp)})$ in the case of $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ theories. To support stability of lumps in those models, we expect that

$$\pi_2(\mathcal{M}_{U(1)\times SO(N)}) \simeq \mathbb{Z} \times \mathbb{Z}_2 \, , \qquad \pi_2(\mathcal{M}_{U(1)\times USp(2M)}) \simeq \mathbb{Z} \, , \tag{9.44}$$

where the $\mathbb{Z}_2$ charge for the $U(1) \times SO(N)$ case is naturally expected, since the corresponding local vortices have their charges due to $\pi_1(U(1) \times SO(N)/\mathbb{Z}_2) = \mathbb{Z} \times \mathbb{Z}_2$ [118]. To determine the homotopy group in these cases is a complicated task since we have to take non-trivial directions of cohomogeneity into account, and a further study of the moduli space of lumps beyond counting dimensions is also needed. This problem still remains as a future problem. The relation between our solutions and the lumps in non-supersymmetric $SO(N)$ QCD [228, 229] is, therefore, unclear so far. In their case, the lumps are supported by the homotopy group $\pi_2[SU(N_{\rm F})/SO(N_{\rm F})] \simeq \mathbb{Z}_2$. Therefore, these lumps are non-BPS. In our case, the gauge coupling constants for $SO(N_{\rm C})$ and $U(1)$ could be different although we did not take that into account. Let g and e be the gauge couplings of the $SO(N_{\rm C})$ and $U(1)$ gauge groups, respectively. We have taken the strong gauge coupling limit for both the couplings, $g, e \to \infty$, in which case the gauge theory reduces to the NLσM of the $U(1) \times SO(N_{\rm C})$ Kähler quotient. Without taking the strong coupling limit for e, the size (width) $1/e\sqrt{\xi}$ for the "Abelian" vortices becomes larger as the $U(1)$ gauge coupling e becomes smaller. In the limit of vanishing e, we expect that they disappear and only non-BPS $\mathbb{Z}_2$ lumps remain. It is important to clarify this point which also remains as a future problem.

Besides these problems, there are many interesting future problems in the following.

Time-dependent stationary solutions, called Q-lumps [253, 254], are also BPS states in a NLσM with a potential. Q-lumps were constructed in the $\mathbb{C}P^1$ model [253, 254], the Grassmann σ model ($U(N_{\rm C})$ Kähler quotient) [255, 256], and the asymptotically Euclidean spaces [80, 81, 30]. It is one of the possible extensions to construct Q-lumps also in $U(1) \times SO(N_{\rm C})$ and $U(1) \times USp(2M_{\rm C})$ Kähler quotients.

Finally, many extensions and applications of the present works include: dynamics of lumps [82, 83], cosmic lump strings [88, 89, 90, 91, 92, 93, 94, 95, 96, 84, 85] and especially their reconnection [112], composite states like triple lump-string intersections [80, 81, 30] and lump-strings stretched between domain walls [29, 98, 99], and the Seiberg-like duality [116].

Part IV

Soliton substructures and fractional vortices and lumps

CHAPTER 10

Fractional vortices and lumps

We study the so-called fractional vortices, i.e. vortex configurations with the minimum winding from the viewpoint of their topological stability, but which are characterized by various notable substructures in the transverse energy distribution. The fractional vortices occur in various Abelian or non-Abelian generalizations of the Higgs model. We identify the two crucial ingredients for their occurrence – the vacuum degeneracy leading to non-trivial vacuum moduli $\mathcal{M}$, and the BPS nature of the vortices and we classify the solutions into two kinds. The first type of such vortices appear when $\mathcal{M}$ has $\mathbb{Z}_n$ orbifold singularities; the second type occurs in systems in which the vacuum moduli space $\mathcal{M}$ possesses some deformed geometry.

10.1 Fractional characteristics and types

We will first consider taking the strong coupling limit, forcing the configuration to stay in the vacuum manifold, $\mathcal{M}$ and we will have in mind generic configurations of the semi-local type, i.e. vortex configurations with non-vanishing size moduli. These configurations will be well-defined in the strong coupling limit, that is, they be lumps.

Even if we choose a regular base point p – vacuum of the theory, the energy distribution in $\mathbb{C}$ feels the structure of the vacuum manifold $\mathcal{M}$ as the volume of the target space is mapped into the transverse plane $\mathbb{C}$ of the semi-local vortex

$$E = 2\int_{\mathbb{C}} \frac{\partial^2 K}{\partial\phi^I \partial\phi^{\dagger\bar{J}}} \partial\phi^I \bar{\partial}\phi^{\dagger\bar{J}} = 2\int_{\mathbb{C}} \bar{\partial}\partial K \ . \tag{10.1}$$

There are mainly two mechanisms leading to multiple peaks in the energy density.

The first is commonly at work in the presence of some orbifold singularities. Then a regular configuration will be such that the wrapping of the target space $\mathcal{M}$ will wind as many times as not to produce a singular configuration. This is in general the product of the orbifold singularities. For instance $\mathbb{Z}_m, \mathbb{Z}_n$ will yield the minimal regular configuration with winding number $n \times m$. When the map from the target space to the configuration space winds several times to avoid the orbifold singularity, this makes the configuration similar to a composite lump in the case of a regular target space $\mathcal{M}$. The difference in this case is that the other objects (position moduli), not having to avoid the orbifold singularity can,

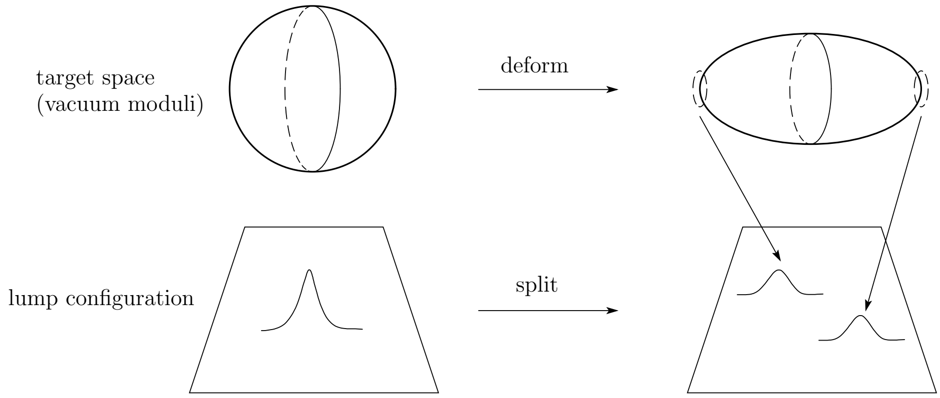


Figure 10.1: A sketch of fractional lumps of the second type in the non-linear σ model (NLσM) limit.

in general, split and produce multiple peaks. Hence, in this case, it indeed is a minimal configuration (under the choice of the base point – VEV), but producing a fractional lump (which becomes a vortex in the finite gauge coupling limit).

The other mechanism at work is somewhat intuitively understood by the fact that the metric appears in formula for the energy density. It is indeed the pull-back of the area form on the target space. We conjecture that in general there will be a correspondence between a positive scalar curvature in the target space $\mathcal{M}$ and a peak in the energy density. This can produce multiple peaks in the presence of multiple areas of (sufficiently large) scalar curvature on $\mathcal{M}$. When the field configuration sweeps such regions, the energy density will show sub-peaks as illustrated in Fig. 10.1[1]. Now we can also understand somewhat intuitively, what happens in the case that there appears a Coulomb singularity on the target space $\mathcal{M}$. However, we need to be cautious now. The strong coupling limit, corresponding to integrating out gauge degree of freedom, is no longer a valid approximation and we should really go back to the full dynamical system. Thus we need to turn on a finite gauge coupling and by continuity, we can understand how, in this case, a fractional vortex has come to life. In turn, we can think of going arbitrarily close to the singularity, for example by turning on a small FI parameter etc. We can now think in terms of the non-linear σ model (NLσM) limit and the mapping produces a regular peak at that point. Of course fractionality in this second type requires a minimum number of active curvatures/singularities to be two.

10.2 The droplet model

We will start by considering the most minimalistic case, namely a $U(1)$ model with $N_{\mathrm{F}} = 2$ flavors and (unequal) charges

[1]The existence of the directions in the target space, which are not related to any isometry, is necessary for the fractional lumps of the second type. Such directions are parametrized by so-called quasi-Nambu-Goldstone modes in the context of supersymmetric theories [160, 257] while the directions of isometries correspond to Nambu-Goldstone modes.

	$U(1)$
A	2
B	1

$$H = \begin{pmatrix} A \\ B \end{pmatrix} . \tag{10.2}$$

The Lagrangian of this model reads

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \left(\mathcal{D}_\mu H\right)^\dagger \mathcal{D}^\mu H - \frac{e^2}{4} \left| \mathrm{Tr}\left(H H^\dagger c\right) - \xi \right|^2 , \tag{10.3}$$

where $\xi > 0$ is an FI-parameter with the covariant derivative

$$\mathcal{D}_\mu H = \left(\partial_\mu \mathbb{1}_2 + \frac{i}{\sqrt{2}} A_\mu c\right) H , \tag{10.4}$$

with the charges $c = \mathrm{diag}(c_A, c_B) = \mathrm{diag}(2, 1)$. Notice the non-standard transposed notation. The gauge transformations take the form,

$$(A, B)^{\mathrm{T}} \to (e^{i2\alpha(x)} A, e^{i\alpha(x)} B)^{\mathrm{T}} . \tag{10.5}$$

The vacuum manifold M (D-flatness condition) is topologically equivalent to S^3 and the vacuum moduli $\mathcal{M}$ are topologically the same as $\mathbb{C}P^1$ but with a conical singularity (see Fig. 10.2)

$$M = \{A, B \quad | \quad 2|A|^2 + |B|^2 = \xi\} , \tag{10.6}$$
$$\mathcal{M} = M/U(1) \simeq W\mathbb{C}P^1_{(2,1)} \simeq \mathbb{C}P^1/\mathbb{Z}_2 . \tag{10.7}$$

The vacuum moduli can be also described by the following quotient

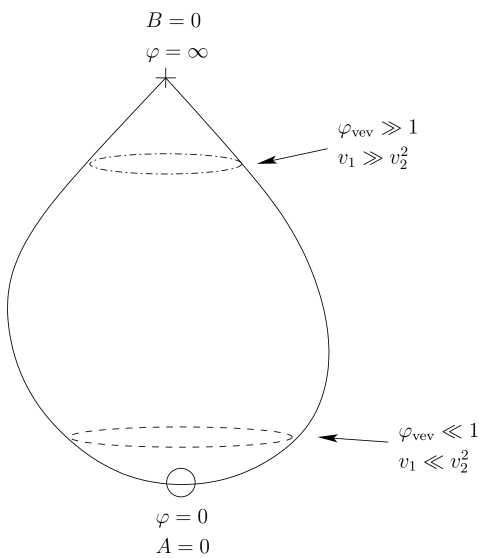


Figure 10.2: A sketch of the $\mathbb{C}P^1/\mathbb{Z}_2$ which is a sphere with a conical singularity at a (i.e., north) pole.

$$(A,B)^{\mathrm{T}} \sim (\lambda^2 A, \lambda B)^{\mathrm{T}}\,, \quad \lambda \in \mathbb{C}^*\,. \tag{10.8}$$

Clearly, $B = 0$ is a $\mathbb{Z}_2$ fixed point. The $U(1)$ gauge symmetry is broken at every point of the vacuum moduli, thus topologically stable vortices can appear.

The Bogomol'nyi completion can be made as follows

$$T = \int_{\mathbb{C}} \left\{ \frac{1}{2e^2} \left| F_{12} - \frac{e^2}{\sqrt{2}} \left(\mathrm{Tr}\left(HH^\dagger c\right) - \xi \right) \right|^2 + 4|\bar{\mathcal{D}}H|^2 - \frac{\xi}{\sqrt{2}} F_{12} - i\varepsilon^{ij}\partial_i \left(H^\dagger \mathcal{D}_j H \right) \right\}\,, \tag{10.9}$$

which determines the BPS-equations and the tension

$$\bar{\mathcal{D}}H = 0\,, \quad F_{12} = \frac{e^2}{\sqrt{2}} \left(\mathrm{Tr}\left(HH^\dagger c\right) - \xi \right)\,, \quad T = -\frac{\xi}{\sqrt{2}} \int_{\mathbb{C}} F_{12} > 0\,. \tag{10.10}$$

We solve the first BPS equation in (10.10) by the Ansatz $H = s^{-1}(z,\bar{z}) H_0(z)$, where

$$H_0(z) = \begin{pmatrix} A_0(z) \\ B_0(z) \end{pmatrix}\,, \tag{10.11}$$

is holomorphic and called the moduli matrix. The field s is determined as

$$\frac{1}{\sqrt{2}} \bar{A} c = -i\bar{\partial} \log\, s\,, \qquad s = \mathrm{diag}(\eta^{c_A}, \eta^{c_B}) = \mathrm{diag}(\eta^2, \eta^1)\,. \tag{10.12}$$

The second BPS equation in (10.10) leads to the master equation

$$\bar{\partial}\partial \log |\eta|^2 = -\frac{e^2}{4} \left[|\eta|^{-2} \left(2|\eta|^{-2} |A_0|^2 + |B_0|^2 \right) - \xi \right]\,, \tag{10.13}$$

where the energy density and tension of the vortex read

$$\mathcal{E} = 2\xi \bar{\partial}\partial \log |\eta|^2 + \partial_i^2 J\,, \quad J \equiv \frac{1}{2} |\eta|^{-2} \left(|\eta|^{-2} |A_0|^2 + |B_0|^2 \right)\,, \tag{10.14}$$

$$T = \int_{\mathbb{C}} \mathcal{E} = 2\pi\xi\nu\,. \tag{10.15}$$

We choose the boundary condition

$$(A,B)^{\mathrm{T}} \to (A_{\mathrm{vev}} e^{i2\nu\theta}, B_{\mathrm{vev}} e^{i\nu\theta})^{\mathrm{T}} \in M \quad \text{as} \quad |z| \to \infty\,. \tag{10.16}$$

We can obtain an analytic solution to the master equation (10.13) in the strong gauge coupling limit $e \to \infty$

$$|\eta|^2 = \frac{|B_0|^2}{2\xi} f(\varphi, \bar{\varphi})\,, \quad f(\varphi,\bar{\varphi}) \equiv 1 + \sqrt{1 + 2|\varphi|^2}\,, \quad \varphi \equiv 2\sqrt{\xi} \frac{A_0}{B_0^2}\,, \tag{10.17}$$

corresponding to a lump solution in the NLσM, whereas the holomorphic field $\varphi(z)$ will turn out to be the inhomogeneous coordinate on the target space of this NLσM.

We consider the minimal-energy vortex configuration ($k = 1$). When we choose a generic point ($B_{\rm vev} \neq 0$) as the boundary condition, the minimal configuration has $U(1)$ winding $\nu = 1$ whose energy is

$$T_{k=1} = 2\pi\xi \ , \quad (\nu = 1) \ . \tag{10.18}$$

The corresponding moduli matrix is given by

$$A_0(z) = A_{\rm vev} z^2 + a_1 z + a_2 \ , \quad B_0(z) = B_{\rm vev} z + b_1 \ , \qquad \{a_1, a_2, b_1\} \in \mathbb{C}^3 \ , \tag{10.19}$$

with $2|A_{\rm vev}|^2 + |B_{\rm vev}|^2 = \xi$. Although this is the minimal-energy configuration, we have three complex moduli parameters a_1, a_2, b_1. Remember that A (B) is zero at a point where A_0 (B_0) is zero. Note that A_0 has two zeros and B_0 has one zero because A winds twice and B winds once when we go around the boundary, S^1 at spatial infinity. An important observation is that the $U(1)$ gauge symmetry is not generally recovered at the zeros; only in the special case where A_0 and B_0 vanish simultaneously.

Consider now the vortex at the special point of the vacuum moduli, $B_{\rm vev} = 0$. The minimal configuration $k = 1$ corresponds to $\nu = 1/2$ and has a tension

$$T^{\rm special}_{k=1} = \pi\xi \ , \quad (\nu = 1/2) \ . \tag{10.20}$$

The moduli matrix now takes the form ($k = 1$)

$$A_0 = A_{\rm vev} z + a \ , \quad B_0 = b \ , \qquad A_{\rm vev} = \sqrt{\xi/2} \ , \tag{10.21}$$

where a, b are the moduli parameters. Comparing this with Eq. (10.19) with $B_{\rm vev} = 0$, one immediately sees that the latter is not a minimal-energy solution.

The vortex (energy, and magnetic flux) profiles can be approximately determined in the strong-coupling limit. The gauge theory reduces to the NLσM whose target space is the vacuum moduli $\mathcal{M}$ in Eq. (10.7). The Kähler potential is given, in the supersymmetric version of our model, by

$$K = |A|^2 e^{-2V} + |B|^2 e^{-V} + \xi V \ . \tag{10.22}$$

Integrating out the $U(1)$ vector multiplet V, we obtain the following Kähler quotient in terms of an inhomogeneous coordinate $\varphi(z)$ which was already found in Eq. (10.17) in the strong gauge coupling limit

$$K = \xi \log f(\varphi, \bar{\varphi}) + \xi f^{-1}(\varphi, \bar{\varphi}) \ , \quad f(\varphi, \bar{\varphi}) \equiv 1 + \sqrt{1 + 2|\varphi|^2} \ . \tag{10.23}$$

Note that the first term is due to the magnetic flux F_{12} and the second term corresponds to the surface term $\partial_i^2 J$ in Eq. (10.15). All the regular BPS lump solutions are given analytically by Eq. (10.17). Only the solutions which have points where A and B simultaneously vanish cannot be seen in this limit, because the $U(1)$ gauge symmetry would remain unbroken there. Such solutions contain small lump singularities and we should go back to the original gauge theory in order to observe the configurations correctly.

The metric on this manifold is

$$g_{\varphi\bar{\varphi}} = \xi \frac{2|\varphi|^2 + f(\varphi,\bar{\varphi})}{f^3(\varphi,\bar{\varphi})(1+2|\varphi|^2)} , \tag{10.24}$$

which is seen not to be singular at any finite value of the inhomogeneous coordinate $\varphi(z)$. The scalar curvature reads

$$R = \frac{4\,[2|\varphi|^2 + (2+|\varphi|^2)\,f(\varphi,\bar{\varphi})]}{\xi\,(1+2|\varphi|^2)^{\frac{3}{2}}} . \tag{10.25}$$

The space is topologically equivalent to $\mathbb{C}P^1$, but there is a $\mathbb{Z}_2$ singularity, which we have mentioned already, and to see this we calculate the Kähler quotient and metric in terms of the inverse inhomogeneous coordinate $\tilde{\varphi}(z) \equiv \varphi^{-1}(z)$

$$K = \frac{\xi}{2} h(\tilde{\varphi},\bar{\tilde{\varphi}}) - \xi \log h(\tilde{\varphi},\bar{\tilde{\varphi}}) , \quad h(\tilde{\varphi},\bar{\tilde{\varphi}}) \equiv \sqrt{|\tilde{\varphi}|^2\,(2+|\tilde{\varphi}|^2)} - |\tilde{\varphi}|^2 , \tag{10.26}$$

The metric in terms of this coordinate reads

$$g_{\tilde{\varphi}\bar{\tilde{\varphi}}} = \frac{1 - h(\tilde{\varphi},\bar{\tilde{\varphi}})}{\sqrt{|\tilde{\varphi}|^2\,(2+|\tilde{\varphi}|^2)}} , \tag{10.27}$$

where $h(|\tilde{\varphi}|^2) \to 0$ for $\tilde{\varphi} \to 0$, however the "1" in the numerator gives rise to a conical singularity. That is, the manifold is a $W\mathbb{C}P^1_{2,1} \simeq \mathbb{C}P^1/\mathbb{Z}_2$. The scalar curvature in these coordinates reads

$$R = -\frac{4\,[h(\tilde{\varphi},\bar{\tilde{\varphi}}) + |\varphi|^2 + 2]}{\xi\,(2+|\varphi|^2)^2\,(h(\tilde{\varphi},\bar{\tilde{\varphi}}) - 1)} . \tag{10.28}$$

A numerical result is shown in Fig. 10.3. As we move in the vacuum moduli space $\mathcal{M}$ by varying the VEVs $A_{\rm vev}, B_{\rm vev}$ (or $\varphi_{\rm vev} \equiv 2\sqrt{\xi}A_{\rm vev}/B_{\rm vev}^2$) and changing the vortex moduli parameters the tension density profile shows varying substructures. Since the zeros of the fields do not imply necessarily the restoration of a $U(1)$ gauge symmetry, the positions of the peaks do not in general coincide with the zeros of the fields A, B. Although it is very complicated to specify the positions of peaks analytically, it is easy to visualize it numerically. In Fig. 10.3, we have shown the zeros of A, B and the peaks. We observe that there are no direct relations between the zeros of fields and the positions of the peaks, except at the two poles, $A_{\rm vev} = 0$ (south pole) and $B_{\rm vev} = 0$ (north pole), of the space $\mathcal{M}$.

An axially symmetric peak appears at the zero $z^{\rm N}$ of $B_0(z)$ in the limit $A_{\rm vev} \to 0$; as $A_{\rm vev}$ departs from 0, it decomposes into two sub-peaks. We cannot remove one of the two sub-peaks by pushing its position to infinity. This feature can easily be observed for large $|\varphi_{\rm vev}| \equiv |2\xi A_{\rm vev}/B_{\rm vev}^2| \gg 1$ where the positions of the two peaks can naturally be approximated by the zeros $z = z_i^{\rm S} (i = 1, 2)$ of $A_0(z)$. (Here $A_{\rm vev}(z_1^{\rm S} + z_2^{\rm S}) = -a_1, A_{\rm vev} z_1^{\rm S} z_2^{\rm S} = a_2$, $B_{\rm vev} z^{\rm N} = -b_1$). The energy density

$$\mathcal{E} = 2\partial\bar{\partial}K = 2g_{\varphi\bar{\varphi}}\partial\varphi\bar{\partial}\bar{\varphi} , \tag{10.29}$$

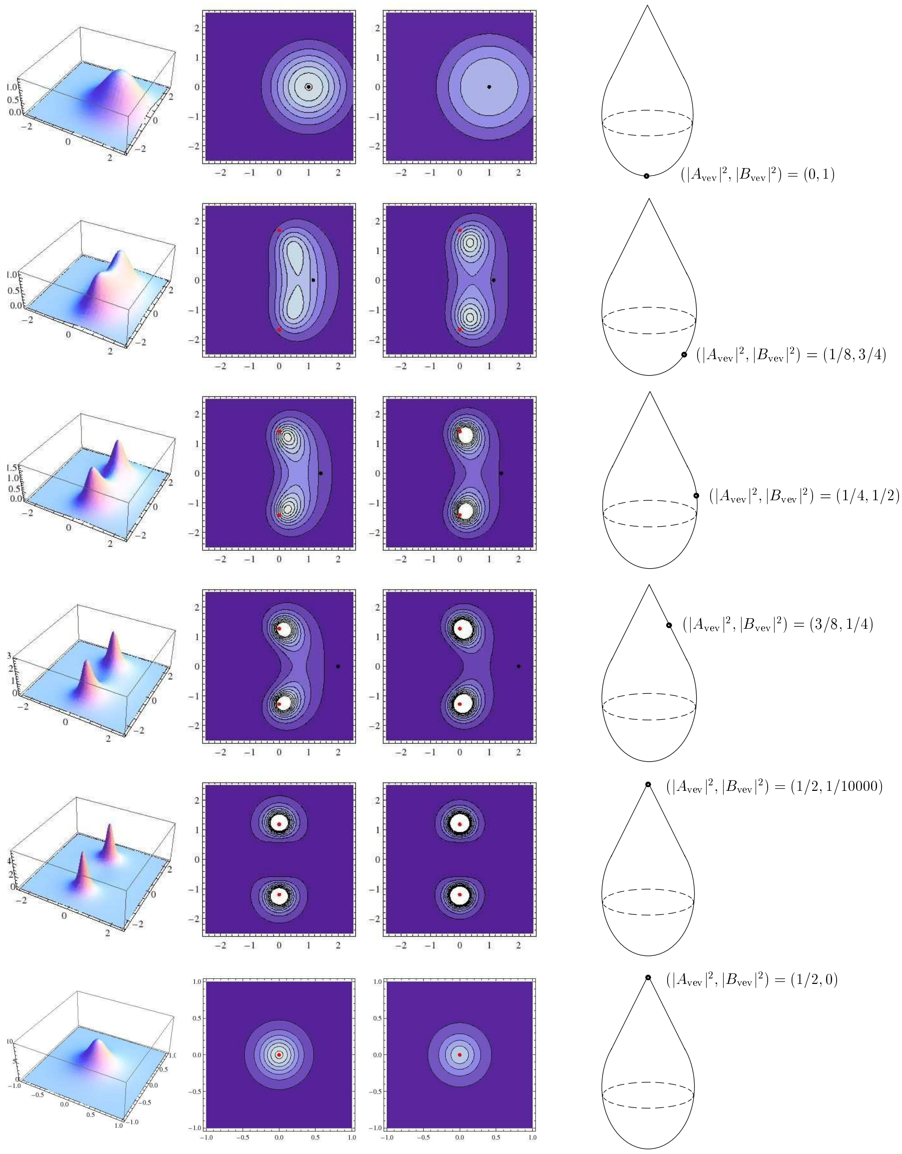


Figure 10.3: The energy (the left-most and the 2nd left panels) and the magnetic flux (the 2nd right panels) density are shown, together with the boundary values (A, B) (the right-most panels) for the minimal lump of the first type in the strong gauge coupling limit. The moduli parameters are fixed as $a_1 = 0, a_2 = 1, b_1 = -1$ in Eq. (10.19). The red dots are zeros of A and the black one is the zero of B. $\xi = 1$. The last figures illustrate the minimum lump defined at exactly the orbifold point (see Eq. (10.21)) with $A_{vev} = 1/\sqrt{2}$, and with $b = 0.8$.

can with the following form of the inhomogeneous coordinate

$$\varphi = \varphi_{\rm vev} \frac{(z - z_1^{\rm S})(z - z_2^{\rm S})}{(z - z^{\rm N})^2} , \quad \varphi_{\rm vev} \equiv 2\sqrt{\xi} \frac{A_{\rm vev}}{B_{\rm vev}^2} , \tag{10.30}$$

be written at the points of the sub-peaks as

$$\mathcal{E}|_{z=z_i^{\rm S}} = \frac{1}{2} \xi |\varphi_{\rm vev}|^2 \frac{|z_1^{\rm S} - z_2^{\rm S}|^2}{|z_i^{\rm S} - z^{\rm N}|^4} . \tag{10.31}$$

For instance, if the three zeros get separated by large distances, then we see that the sub-peaks are diluted. If, instead, only one of the zeros, $z = z_2^{\rm S}$ is pushed toward infinity, that is $|z_2^{\rm S} - z^{\rm N}|, |z_1^{\rm S} - z_2^{\rm S}| \gg |z_1^{\rm S} - z^{\rm N}|$, the peak at $z = z_1^{\rm S}$ becomes singular. In either case, a single isolated peak is not allowed as a vortex (lump) solution. This solution consisting of two sub-peaks is one of the typical examples of fractional vortices. Only when $B_{\rm vev} = 0$, they become independent. Such a limiting configuration is no longer a minimal energy configuration, however. The minimal configuration at exactly $B_{\rm vev} = 0$ (with (10.21)) has only one peak. Its tension is half of the minimal configuration for $B_{\rm vev} \neq 0$.

The reason why the minimum vortex at $B_{\rm vev} \neq 0$ must have twice the energy with respect to the minimal object at $B_{\rm vev} = 0$ is as follows. Our vacuum manifold has a $\mathbb{Z}_2$ singularity at $|B_{\rm vev}| = 0$. If the vacuum is chosen at $B_{\rm vev} \neq 0$ the solution touches the singularity at a finite point in the z-plane and would get singular there. To remove such a singularity, the solution must wrap twice around the vacuum moduli. On the other hand, if one sits exactly at the $\mathbb{Z}_2$ point of the manifold the solution does not touch the singularity within a finite distance and a regular solution can be constructed with just a single winding.

A comment in store is about the "volume" of the sub-peaks. Roughly speaking, in this simple model, the energy contribution being mapped to the peaks corresponding to the moduli $z_i^{\rm S}, i = 1, 2$ comes from the pull-back of the area form from the value of the VEV, namely $\varphi_{\rm vev}$, over the south-pole and back. Similarly the contribution for the peak corresponding to the moduli $z^{\rm N}$ comes from the integral over the north-pole.

Now we consider directly the NLσM and write the three minimal configurations as follows

$$\varphi^{[11]}(z) = \varphi_{\rm vev} \frac{(z - z_1^{\rm S})(z - z_2^{\rm S})}{(z - z^{\rm N})^2} , \quad \varphi^{[01]}(z) = \frac{z - z_1^{\rm S}}{(z - z^{\rm N})^2} , \quad \varphi^{[10]}(z) = z - z_1^{\rm S} . \tag{10.32}$$

The first solution $\varphi^{[11]}$ has a generic VEV, while $\varphi^{[01]}$ and $\varphi^{[10]}$ will go to 0 and ∞ for $|z| \to \infty$, respectively. It is evident that there are more possibilities in the NLσM only. However, if we want to make sense to the model in terms of the gauge theory at finite gauge coupling, it is easy to see that all poles have to have a multiplicity of an even number. Considering splitting the pole into two poles of multiplicity one

$$\varphi(z) = \varphi_{\rm vev} \frac{(z - z_1^{\rm S})(z - z_2^{\rm S})}{(z - z_1^{\rm N})(z - z_2^{\rm N})} , \tag{10.33}$$

we can no longer turn on a finite gauge interaction in this model.

The characteristics of the vortex-energy profile are deeply rooted in the property of the vacuum manifold $\mathcal{M}$ itself and to its singularity structure. In a mathematical language, the wrapping of target space of the NLσM in this simple example makes a discontinuous jump at $\varphi_{\rm vev} = \infty$ ($B_{\rm vev} = 0$)

$$\frac{\pi_2\left(\mathcal{M}, \varphi_{\rm vev}\right)}{\pi_2\left(\mathcal{M}, \infty\right)} = \mathbb{Z}_2 \;, \qquad \frac{\pi_1\left(F, f\right)}{\pi_1\left(F, f_0\right)} = \mathbb{Z}_2 \;, \qquad \varphi_{\rm vev} \neq \infty \;, \tag{10.34}$$

where the fiber is the $U(1)$ fibration in the semi-local vortex system. The S^1 fiber itself reduces to the half at the orbifold singularity

$$f = \pi^{-1}(\varphi_{\rm vev}) = S^1 \;, \qquad f_0 = \pi^{-1}(\infty) = S^1/\mathbb{Z}_2 \;. \tag{10.35}$$

This is the global reason for the two sub-peaks observed in Fig. 10.3.

The argument here can be easily extended to more general cases with the multiple flavors $H = (A, B, C, D, \cdots)$ with generic $U(1)$ charges $Q = (m, n, o, q, \cdots)$, which are all relatively prime. The moduli manifold is then $\mathcal{M} = \mathbb{C}P^{N_{\rm F}-1}_{(m,n,\cdots)} \simeq \mathbb{C}P^{N_{\rm F}-1}/(\mathbb{Z}_m \times \mathbb{Z}_n \times \cdots)$. Near a $\mathbb{Z}_m$ singular point, $(|A_{\rm vev}|, |B_{\rm vev}|, |C_{\rm vev}|, \cdots) = (\sqrt{\xi/m}, 0, 0, \cdots,)$ m peaks appear in the energy distribution.

10.2.1 Generalizations

We will now consider a generalization of the model studied in the last Section. We still restrict ourselves to $N_{\rm F} = 2$ flavors $H = (A, B)^{\rm T}$ but with unequal charges assigned as $(\{m, n\})$ to the fields A and B, respectively. Let us assume that m, n are relatively prime.

The vortex solution is characterized by the broken $U(1)$-winding number ν given in Eq. (10.15). Analogously to the previous case, we can write the solutions in terms of the moduli matrix

$$H = (A, B)^{\rm T} = \left(\eta^{-m} A_0(z) \;,\; \eta^{-n} B_0(z)\right)^{\rm T} \;, \tag{10.36}$$

where ν is a positive number, η is an everywhere non-zero function and *the moduli matrices* $A_0(z)$ and $B_0(z)$ are polynomial functions of z. The asymptotic behavior of η is still

$$|\eta|^2 \to |z|^{2\nu} \qquad \text{as} \quad |z| \to \infty \;, \tag{10.37}$$

while the boundary condition now reads

$$(A, B)^{\rm T} \to (A_{\rm vev} e^{im\nu\theta}, B_{\rm vev} e^{in\nu\theta})^{\rm T} \in M \qquad \text{as} \quad |z| \to \infty \;. \tag{10.38}$$

The BPS equations lead to the master equation

$$\bar\partial\partial \log\omega = -\frac{e^2}{4}\left[m\,\omega^{-m}|A_0|^2 + n\,\omega^{-n}|B_0|^2 - \xi\right] \;, \tag{10.39}$$

where $\omega \equiv \eta\eta^\dagger$.

If we fix $A \equiv 0$ ($B \equiv 0$) everywhere, we can think of the system as just the Abelian-Higgs model with one complex scalar field B (A) whose $U(1)$ charge is n (m). The vortices there are the normal ANO solutions, though the k-vortex solutions will have the $U(1)$-winding number k/n (k/m) with tension $T_B = 2\pi\xi k/n$ ($T_A = 2\pi\xi k/m$). Indeed, when only one field is active while the other is inert, the $U(1)$ gauge coupling constant and the FI term can be rescaled such that the system looks exactly as the standard Abelian-Higgs model with unit $U(1)$ charge. What we are trying to study in this Section is an intermediate situation between two kinds of vortices where both fields contribute non-trivially. Such intermediate states should have the energy $T \equiv mT_A = nT_B$, and we shall see configurations which have m peaks in one limit and n peaks in another limit.

First we choose a generic point such as $A_{\rm vev} \neq 0$ and $B_{\rm vev} \neq 0$. The moduli matrices behave asymptotically as follows

$$\begin{aligned} A_0(z) &= \eta^m A \to |z|^{m\nu} e^{im\nu\theta} A_{\rm vev} \, , \\ B_0(z) &= \eta^n B \to |z|^{n\nu} e^{in\nu\theta} B_{\rm vev} \, , \quad \text{as} \quad |z| \to \infty \, . \end{aligned} \tag{10.40}$$

Holomorphy of A_0, B_0 requires $m\nu \in \mathbb{Z}_+$ and $n\nu \in \mathbb{Z}_+$. As we have chosen m and n to be relatively prime, this is satisfied by $\nu \equiv k \in \mathbb{Z}_+$. Thus we have obtained the non-trivial condition for A_0, B_0

$$\nu = k : \quad A_0(z) = A_{\rm vev} z^{mk} + \mathcal{O}(z^{mk-1}) \, , \quad B_0(z) = B_{\rm vev} z^{nk} + \mathcal{O}(z^{nk-1}) \, . \tag{10.41}$$

Note that k vortices have $(m+n)k$ moduli parameters with the boundary vacuum modulus. They may correspond to positions and sizes of the fractional vortices.

When we choose the special point $A_{\rm vev} = 0$ ($\mathbb{Z}_m$ fixed point) or $B_{\rm vev} = 0$ ($\mathbb{Z}_n$ fixed point) as a boundary condition, the conditions for the moduli matrix drastically change. For instance for $|A_{\rm vev}| = \sqrt{\xi/m}$ and $B_{\rm vev} = 0$, we immediately obtain $\nu = k/m$ and the conditions

$$\nu = \frac{k}{m} : \quad A_0 = \sqrt{\frac{\xi}{m}}\, z^k + \mathcal{O}(z^{k-1}) \, , \quad B_0 = b z^\beta + \mathcal{O}(z^{\beta-1}) \, , \tag{10.42}$$

where β is a semi-positive definite integer less than $n\nu = \frac{n}{m}k$. If we set $B_0 = 0$, the solution is identical to the ANO vortex as we mentioned before. When B_0 is not zero, the solutions significantly differ from the ANO solution and also from the semi-local vortices in the extended Abelian-Higgs model. Similarly, if we choose $|B_{\rm vev}| = \sqrt{\xi/n}$ and $A_{\rm vev} = 0$, the $U(1)$ winding number becomes $\nu = k/n$ and the conditions change as

$$\nu = \frac{k}{n} : \quad A_0 = a z^\alpha + \mathcal{O}(z^{\alpha-1}) \, , \quad B_0 = \sqrt{\frac{\xi}{n}}\, z^k + \mathcal{O}(z^{k-1}) \, , \tag{10.43}$$

where α is a semi-positive definite integer less than $m\nu = \frac{m}{n}k$. Note that the $U(1)$ charge ν is fractionally quantized at the conical singularities. The present model thus nicely illustrates the first mechanism for the fractional vortices discussed in Sec. 10.1.

10.2.2 Cousins

Let us make some comments about the many models that were found and studied in the early days of the fractional vortices. It was found that several models with different matter content had exactly the target space of the droplet model (10.7). Naturally, with the arguments of the last Sections, also these models provide fractionality. Instead of describing them in detail here, let us just demonstrate their similarity by their Kähler quotients. For more details the reader can delve into the paper [2].

The first example is the model with the following fields and charges

	$U_1(1)$	$U_2(1)$
A	2	0
B	1	1
C	1	-1

The Kähler potential of this model is given by

$$K = |A|^2 e^{-2V_1} + |B|^2 e^{-V_1} e^{-V_2} + |C|^2 e^{-V_1} e^{V_2} + \xi V_1 \,, \tag{10.44}$$

where the Kähler quotient is given by Eq. (10.23) with the inhomogeneous coordinate

$$\varphi = \sqrt{\xi} \frac{A_0(z)}{B_0(z) C_0(z)} \,. \tag{10.45}$$

In this case the number of moduli is only two.

The next example is the model summarized by

	$U(1)$	$SU(2)$
$\mathcal{A}$	2	$\mathbb{1}$
H	1	$\square$

which has the Kähler potential

$$K \;\; = \mathrm{Tr}\left[H H^\dagger e^{-V'} e^{-V_e} \right] + |\mathcal{A}|^2 e^{-2V_e} + \xi V_e \,. \tag{10.46}$$

The Kähler quotient is again given by Eq. (10.23) but now with the inhomogeneous coordinate

$$\varphi = \sqrt{\xi} \frac{\mathcal{A}_0(z)}{\det H_0(z)} \,. \tag{10.47}$$

In this case the number of moduli is only three. The moduli space in this case reads

$$\mathcal{M} = \mathbb{C} \times \mathbb{C} \times \mathbb{C}P^1 \,. \tag{10.48}$$

10.3 The sweet potato / lemon space

The model of Sec. 10.2 was chosen to have fewest possible fields. However, the Kähler metric turned out to be rather elaborate. A simplification is easily made by choosing equal $U(1)$ charges (up to a sign) for every $U(1)$ gauge group. The simplest example in this genre is summarized by

	$U_1(1)$	$U_2(1)$
A	1	0
B	1	1
C	1	-1

The field content transforms under the gauge symmetry as

$$(A, B, C)^{\mathrm{T}} \to \left(e^{i\alpha(x)}A,\ e^{i\alpha(x)+i\beta(x)}B,\ e^{i\alpha(x)-i\beta(x)}C\right)^{\mathrm{T}} . \tag{10.49}$$

An important difference from the model described briefly in Sec. 10.2.2 is that the gauge symmetry is now $U(1)_1 \times U(1)_2$, without a $\mathbb{Z}_2$ division. The points $(A,\ B\ ,\ C)$ and $(-A,$ $B,\ C)$ related by $(\alpha, \beta) = (\pi, \pm\pi) \in \mathbb{Z}_2$, are distinct points. The vacuum manifold and vacuum moduli space are given by

$$M \;=\; \left\{A, B, C \quad | \quad |A|^2 + |B|^2 + |C|^2 = \xi_1,\ |B|^2 - |C|^2 = \xi_2\right\} , \tag{10.50}$$

$$\mathcal{M} \;=\; M/\left(U(1)_1 \times U(1)_2\right) . \tag{10.51}$$

We see that $A = 0$ is a $\mathbb{Z}_2$ orbifold point, whereas the point $B = C = 0$ represents a system in Coulomb phase (which can be Higgsed and regularized by $\xi_2 \neq 0$). See Fig. 10.4. Clearly this model shares aspects both of the simple $U(1)$ model of Sec. 10.2 and of the $U(1)\times U(1)$ model described in Sec. 10.2.2. In the following we shall consider mainly the case of $\xi_2 = 0$, except when we consider the NLσM limit, which is well defined only for a non-vanishing ξ_2.

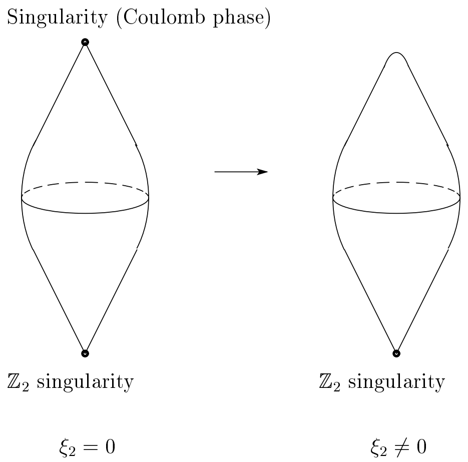


Figure 10.4: The sweet potato / lemon space.

The vortex Ansatz is

$$(A, B, C)^{\mathrm{T}} = \left(s_1^{-1}A_0(z),\ s_1^{-1}s_2^{-1}B_0(z),\ s_1^{-1}s_2C_0(z)\right)^{\mathrm{T}} , \tag{10.52}$$

with the master equations

$$\bar{\partial}\partial \log \omega_1 = -\frac{e^2}{4}\left[\omega_1^{-1}\left(|A_0|^2 + \omega_2^{-1}|B_0|^2 + \omega_2|C_0|^2\right) - \xi_1\right] , \tag{10.53}$$

$$\bar{\partial}\partial \log \omega_2 = -\frac{g^2}{4}\left[\omega_1^{-1}\left(\omega_2^{-1}|B_0|^2 - \omega_2|C_0|^2\right) - \xi_2\right] , \tag{10.54}$$

where $\omega_i \equiv s_i s_i^\dagger$, for $i = 1, 2$. The equations work out analogously as demonstrated in the previous Sections and we will not repeat them here. The winding numbers are ν_1 and ν_2 for $U(1)_1$ and $U(1)_2$, respectively. The tension depends only on ν_1 for $\xi_2 = 0$

$$T = 2\pi\xi_1\nu_1 \ , \quad \nu_1 \in \mathbb{Z}_+ \ . \tag{10.55}$$

The minimal-energy solutions with the generic boundary condition ($0 < |A_{\rm vev}|^2 < \xi_1$) have $T = 2\pi\xi_1$ and are obtained by the following three different moduli matrices

$$\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} A_{\rm vev} z + a \\ B_{\rm vev} \\ C_{\rm vev} z^2 + c_1 z + c_2 \end{pmatrix} , \qquad (\nu_1, \nu_2) = (1, -1) \ , \tag{10.56}$$

$$\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} A_{\rm vev} z + a \\ B_{\rm vev} z + b \\ C_{\rm vev} z + c \end{pmatrix} , \qquad (\nu_1, \nu_2) = (1, 0) \ , \tag{10.57}$$

$$\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} A_{\rm vev} z + a \\ B_{\rm vev} z^2 + b_1 z + b_2 \\ C_{\rm vev} \end{pmatrix} , \qquad (\nu_1, \nu_2) = (1, 1) \ . \tag{10.58}$$

As they obey different boundary conditions for ν_2, they belong to different topological sectors. Each configuration has three moduli parameters.

Near the $\mathbb{Z}_2$ orbifold point we observe two peaks. Although the energy density always looks the same, the magnetic fluxes, especially of the second $U(1)_2$, depends on the value of ν_2. In Fig. 10.5, we show several numerical solutions for Eq. (10.58). We also show a couple of solutions for Eq. (10.57) in Fig. 10.6. In almost all regions, the configuration consists of one peak or two peaks but sometimes we observe three peaks simultaneously.

On the other hand, at exactly the singular vacuum $A_{\rm vev} = 0$ (the singular point on $\mathcal{M}$), the minimal vortex with tension $T = \pi\xi_1$ is given by

$$\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} a \\ B_{\rm vev} z + b \\ C_{\rm vev} \end{pmatrix} , \qquad (\nu_1, \nu_2) = (1/2, 1/2) \ , \tag{10.59}$$

$$\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} a \\ B_{\rm vev} \\ C_{\rm vev} z + c \end{pmatrix} , \qquad (\nu_1, \nu_2) = (1/2, -1/2) \ . \tag{10.60}$$

At $A = 0$ ($\varphi = 0$) a $\mathbb{Z}_2$ symmetry remains unbroken which is a typical orbifold singularity. As a result, the $U(1)_1$ fiber F is the half ($\alpha = 0 \to \pi$) at the orbifold point as compared to that in other points of the vacuum moduli, where $\alpha = 0 \to 2\pi$. The global structure of the vortex-sigma model lumps in this model is thus somewhat similar to the model of Sec. 10.2. At the $\mathbb{Z}_2$ orbifold singularity $\pi_1(F)$ and $\pi_2(\mathcal{M})$ make a jump, and this explains the appearance of the two sub-peaks.

As in the model in Sec. 10.2.2, we cannot take $B_{\rm vev} = C_{\rm vev} = 0$ as a boundary condition since the second $U(1)_2$ is unbroken at infinity.

These aspects can be made more explicit in the strong gauge coupling limit $e_1, e_2 \to \infty$, where the Kähler potential is simpler (than in the model of Sec. 10.2.2) by construction.

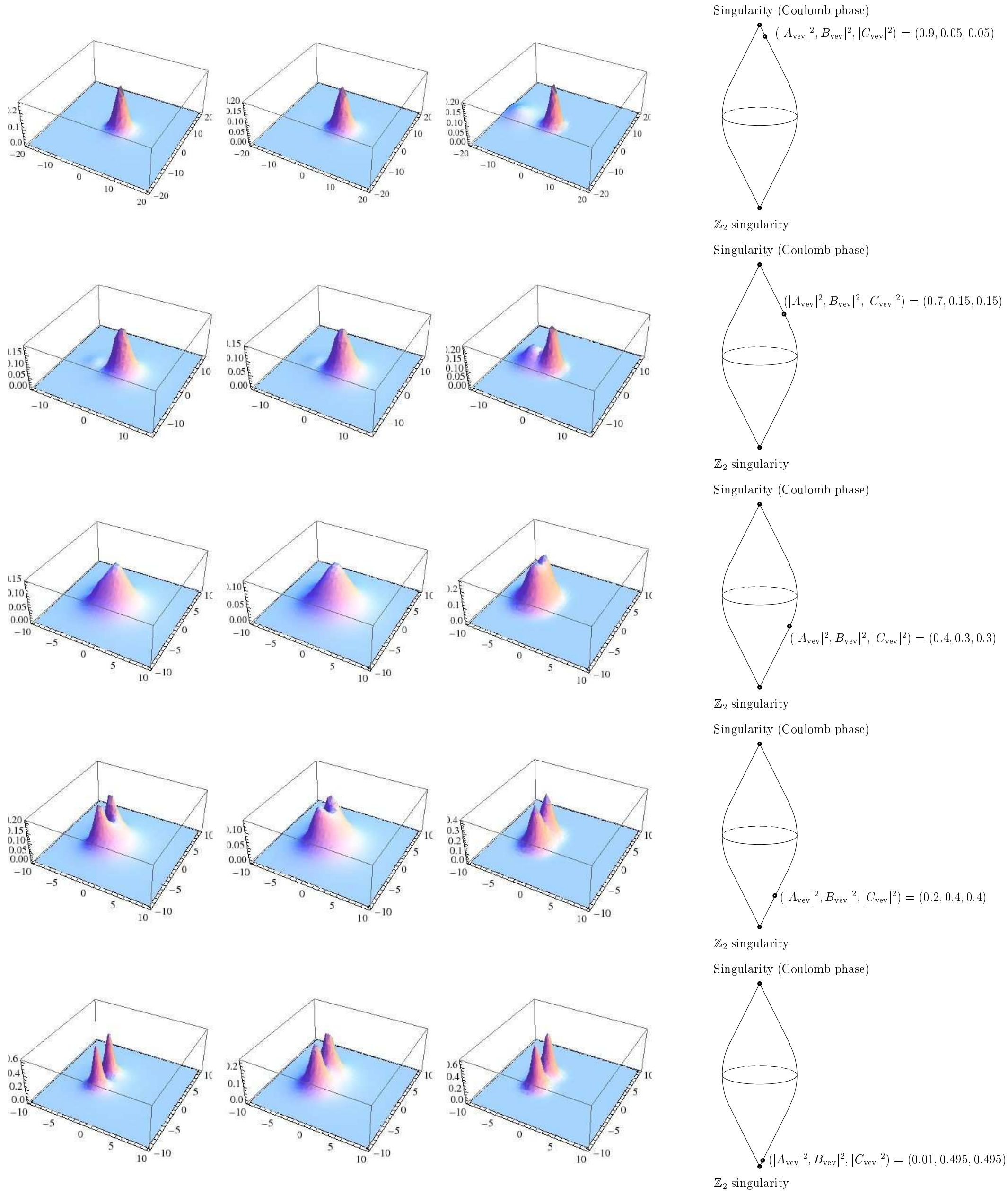


Figure 10.5: The energy density (left-most) and the magnetic flux density $F_{12}^{(1)}$ (2nd from the left), $F_{12}^{(2)}$ (2nd from the right) are shown together with the boundary condition (right-most) for Eq. (10.58) with $\xi_1 = 1$ and $\xi_2 = 0$ and $e_1 = 1$, $e_2 = 2$.

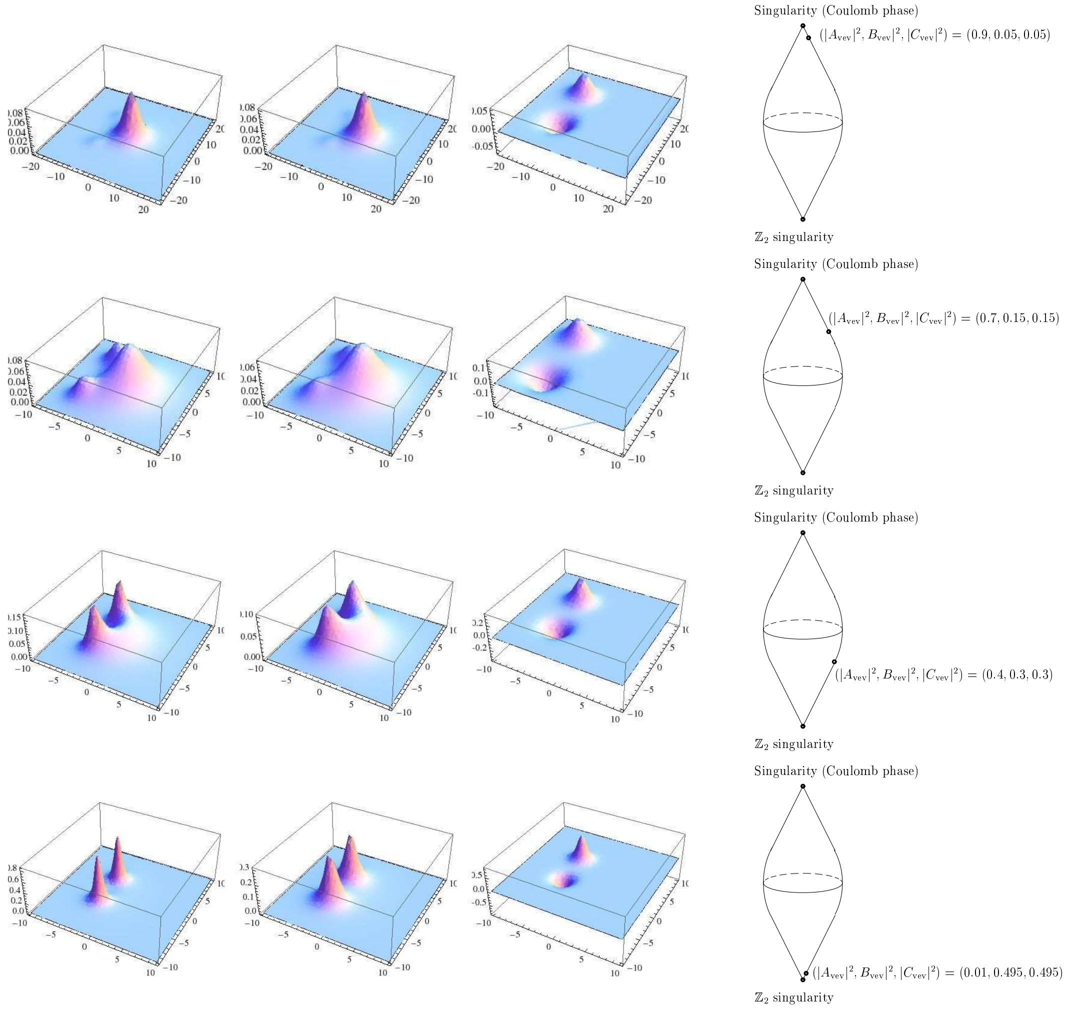


Figure 10.6: The energy density (left-most) and the magnetic flux density $F_{12}^{(1)}$ (2nd from the left), $F_{12}^{(2)}$ (2nd from the right) are shown together with the boundary condition (right-most) for Eq. (10.57) with $\xi_1 = 1$ and $\xi_2 = 0$ and $e_1 = 1$, $e_2 = 2$.

Since the Coulomb phase leads to singular solutions, we here turn on the another FI parameter ξ_2 ($|\xi_2| < \xi_1$) for $U(1)_2$. Working in a supersymmetric context, integrating out the gauge superfields V_1 and V_2 from

$$K = |A|^2 e^{-V_1} + |B|^2 e^{-V_1 - V_2} + |C|^2 e^{-V_1 + V_2} + \xi_1 V_1 + \xi_2 V_2 \,, \tag{10.61}$$

yields the Kähler quotient

$$K = \xi_1 \log \left(1 + \sqrt{\lambda^2 + (1-\lambda^2)|\tilde{\varphi}|^2} \right) - |\xi_2| \log \left(|\lambda| + \sqrt{\lambda^2 + (1-\lambda^2)|\tilde{\varphi}|^2} \right) \,. \tag{10.62}$$

where $\lambda \equiv \frac{\xi_2}{\xi_1}$ and $\tilde{\varphi}$ is the inhomogeneous coordinate which describes the BPS solutions in terms of the holomorphic function

$$\tilde{\varphi}(z) = \frac{2B_0(z)C_0(z)}{A_0^2(z)} \,, \tag{10.63}$$

and the solutions are characterized by the quantized tension

$$T = 2\pi \sum_i \xi_i \nu_i \,, \qquad \nu_i \equiv \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\partial \log |s_i|^2 \,, \tag{10.64}$$

where $|s_i|^2$ can be solved analytically and are given by

$$|s_1|^2 = \frac{|A_0|^2}{1-\lambda^2}\left(1+\sqrt{\lambda^2+(1-\lambda^2)|\tilde{\varphi}|^2}\right) \,, \tag{10.65}$$

$$|s_2|^2 = \frac{|A_0|^2}{|C_0|^2}\frac{-\lambda+\sqrt{\lambda^2+(1-\lambda^2)|\tilde{\varphi}|^2}}{2(1+\lambda)} = \frac{|B_0|^2}{|A_0|^2}\frac{2(1-\lambda)}{\lambda+\sqrt{\lambda^2+(1-\lambda^2)|\tilde{\varphi}|^2}} \,. \tag{10.66}$$

10.3.1 Another cousin

As in the case of the model discussed in Sec. 10.2, there has also been found a cousin of this model with the sweet potato / lemon space as a target space of the NLσM, however with $\xi_2 = 0$. Again more details about this model can be found in the paper [2]. The model is summarized by the following matter and charge assignments

	$U(1)$	$SU(2)$
$\mathcal{A}$	1	$\mathbb{1}$
$\mathbf{B}$	1	$\square$

The Kähler potential is

$$K = \mathrm{Tr}\left[HH^\dagger e^{-V'}e^{-V_e}\right] + |\mathcal{A}|^2 e^{-V_e} + \xi V_e \,, \tag{10.67}$$

which has the Kähler quotient (10.62) with $\lambda = \xi_2 = 0$, i.e.

$$K = \xi_1 \log\left(1+\sqrt{|\tilde{\varphi}|^2}\right) \,, \tag{10.68}$$

and the inhomogeneous coordinate given by

$$\varphi(z) = 2\sqrt{\xi_1}\frac{\det H_0}{\mathcal{A}_0} \,. \tag{10.69}$$

In this case the moduli space is

$$\mathcal{M} = \mathbb{C}\times\mathbb{C}\times\mathbb{C}P^1 \,. \tag{10.70}$$

10.4 $U(1)\times SO(N)$ model

We now consider the fractional vortices occurring in a model with gauge group $U(1)\times SO(N)$. This was the first fractional vortex (lump) that was discovered [5].

For our purposes here, we shall consider only the even-dimensional orthogonal groups, i.e. $N = 2M$. The matter content is $N_{\mathrm{F}} = N$ flavors of squarks in the fundamental representation of the $SO(N)$ group, all with the same unit charge with respect to the $U(1)$ group:

	$U(1)$	$SO(N)$
H	1	$\square$

This theory has been studied in depth in this thesis in various Chapters, both with respect to the vacua, the vortices and the corresponding lumps. Hence we will not repeat any unnecessary equations here.

In order to study the minimal winding vortex configuration concretely, we will consider the following moduli matrix

$$H_0 = \begin{pmatrix} z\mathbf{1}_M - \mathbf{Z} & \mathbf{C} \\ 0 & \mathbf{1}_M \end{pmatrix} , \quad \mathbf{Z} = \mathrm{diag}(z_1, z_2, \ldots, z_M) , \quad \mathbf{C} = \mathrm{diag}(c_1, c_2, \ldots, c_M) , \tag{10.71}$$

and first restrict ourselves to the standard color-flavor locked vacuum (1.40). However, the vacuum is generically given by Eq. (8.4). To solve the master Eqs. (1.58) and (1.59), we set

$$\Omega' = \mathrm{diag}\left(e^{\chi'_1}, \ldots, e^{\chi'_M}, e^{-\chi'_1}, \ldots, e^{-\chi'_M}\right) , \tag{10.72}$$

where the determinant one is manifest. Taking $\omega = e^{\psi}$, we obtain

$$\bar{\partial}\partial\psi = -\frac{e^2}{8M}\left[\sum_{i=1}^{M}\left\{\left(|z-z_i|^2 + |c_i|^2\right)e^{-(\psi+\chi'_i)} + e^{-(\psi-\chi'_i)}\right\} - \xi\right] , \tag{10.73}$$

$$\bar{\partial}\partial\chi'_i = -\frac{g^2}{8}\left[\left(|z-z_i|^2 + |c_i|^2\right)e^{-(\psi+\chi'_i)} - e^{-(\psi-\chi'_i)}\right] , \qquad \forall\, i \in [1, M] . \tag{10.74}$$

If we now take the infinite gauge coupling limit $e \to \infty,\ g \to \infty$, we obtain the following lump solution

$$e^{\chi'_i} = \sqrt{|z-z_i|^2 + |c_i|^2} , \tag{10.75}$$

$$e^{\psi} = \frac{2}{\xi}\sum_{i=1}^{M} e^{\chi'_i} = \frac{2}{\xi}\sum_{i=1}^{M}\sqrt{|z-z_i|^2 + |c_i|^2} , \tag{10.76}$$

which has the energy density

$$\mathcal{E} = 2\xi\bar{\partial}\partial\log\left\{\sum_{i=1}^{M}\sqrt{|z-z_i|^2 + |c_i|^2}\right\} . \tag{10.77}$$

This is the fractional lump solution found in Ref. [5].

The vortex energy profile in the strong-coupling approximation for the $U(1) \times SO(6)$ model is shown in Fig. 10.7. Three fractional peaks are clearly seen. The positions of the peaks can be understood as follows. If $c_i = 0$ one of the $\hat{U}(1) \subset U(1) \times SO(2M)$, constructed as the diagonal combination of $U(1)$ and one of the $U(1)$ Cartan subalgebra of $SO(2M)$, is restored at the points $z = z_i$ ($i = 1, 2, \ldots M$). If $c_i \neq 0$ the situation around a fractional peak at $z = z_i$ is similar to the power-behaved semi-local vortex of the extended Abelian-Higgs model. The number of peaks reflects obviously the rank of the

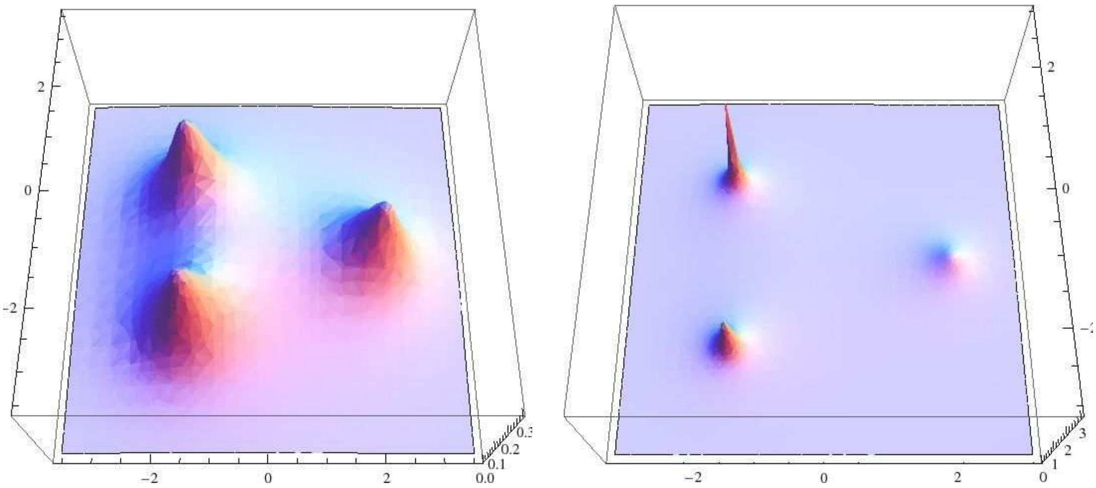


Figure 10.7: The energy density of three fractional vortices (lumps) in the $U(1) \times SO(6)$ model in the strong coupling approximation. The positions are $z_1 = -\sqrt{2} + i\sqrt{2}, z_2 = -\sqrt{2} - i\sqrt{2}, z_3 = 2$. *Left panel*: the size parameters are chosen as $c_1 = c_2 = c_3 = 1/2$. *Right panel*: the size parameters are chosen as $c_1 = 0, c_2 = 0.1, c_3 = 0.3$. Notice that one peak is singular (z_1) and the other two are regularized by the finite (non-zero) parameters $c_{2,3}$.

group considered (here $\mathrm{rank}\{SO(6)\} = 3$), but the number of the possible fractional peaks depends on the point of the vacuum moduli (a particular VEV) considered. For instance, if two of v_i are taken to be zero, the maximum number of the fractional peaks would be two, and so on.

In the supersymmetric version of the models based on the $U(1) \times SO(N)$ gauge groups, the Kähler potential in terms of a meson M has been determined in Ref. [5],

$$K = \xi \log \mathrm{Tr}\sqrt{MM^\dagger} \,. \tag{10.78}$$

If we relax the vacuum moduli to be equal (1.40), thus having the possibility of distinct $\{a_i\}$'s in Eq. (8.4), it will prove convenient to work directly with the mesons of $SO(2M)$

$$M = \begin{pmatrix} e^u(z-\alpha) & \pm i\alpha \\ \pm i\alpha & e^{-u}(z+\alpha) \end{pmatrix} \,, \tag{10.79}$$

with $\alpha, u \in \mathbb{R}$. The meson VEV will be $\mathrm{diag}(a_1^2, a_2^2) = \mathrm{diag}(e^u, e^{-u})$. Using the Kähler potential (10.78) we readily obtain the energy density

$$\mathcal{E} = \xi\bar{\partial}\partial \log\left(|z - \alpha\tanh(u)|^2 + \frac{\alpha^2}{\cosh^2(u)}\right) \,. \tag{10.80}$$

Furthermore, we can construct a typical example of fractional vortices, in a $U(1) \times SO(2N)$ model in the lump limit as follows

$$\mathcal{E} = 2\xi\bar{\partial}\partial \log\left(\sum_{i=1}^{N} m_i \sqrt{|z - \alpha_i\tanh(u_i)|^2 + \frac{\alpha_i^2}{\cosh^2(u_i)}}\right) \,, \tag{10.81}$$

with $a_{2i-1}^2 = m_i\, e^{u_i}, a_{2i}^2 = m_i\, e^{-u_i}$. For each $SO(2)$ subgroup, we have in this construction a possibility for an amplification m_i, while α_i, u_i serve as a position- and an effective size-parameters. One can observe that m_i controls the relative weight of the energy distributed to the i-th fractional vortex.

10.5 Discussion

In this Chapter we have given a simple account of the fractional vortices which have minimally quantized magnetic flux (winding) but with non-trivial substructures in the energy distribution in the transverse plane. The common characteristic features these models share are a non-trivial vacuum degeneracy and the BPS saturated nature of the vortex solutions. We have generalized the moduli matrix formalism [29, 100, 136] to clarify all possible moduli parameters of the minimally quantized fractional vortices.

We have classified fractional vortices into two types; the first type appears when $\mathcal{M}$ has a $\mathbb{Z}_n$ singularity where the gauge symmetry is not restored while the second type occurs when $\mathcal{M}$ has a 2-cycle with a deformed geometry. Indeed, we have observed that smooth fractional lump solutions become singular as the smooth manifold $\mathcal{M}$ is deformed into a singular manifold (e.g. when some FI parameters are turned off). Even when $\mathcal{M}$ has such singularities, we have found smooth fractional vortex solutions. The vortices share the same properties as those of the corresponding lumps wrapping on $\mathcal{M}$ smoothened.

Part V

Non-Abelian Chern-Simons vortices

CHAPTER 11

Non-Abelian Chern-Simons vortices with generic gauge groups

We study non-Abelian Chern-Simons BPS-saturated vortices enjoying $\mathcal{N} = 2$ supersymmetry in $d = 2 + 1$ dimensions, with generic gauge groups of the form $U(1) \times G'$, with G' being a simple group, allowing for orientational modes in the solutions. We will keep the group as general as possible and utilizing the powerful moduli matrix formalism to provide the moduli spaces of vortices and derive the corresponding master equations. Furthermore, we study numerically the vortices applying a radial Ansatz to solve the obtained master equations and we find especially a splitting of the magnetic fields, when the coupling constants for the trace-part and the traceless part of the Chern-Simons term are varied, such that the Abelian magnetic field density can become negative near the origin of the vortex while the non-Abelian part stays positive, and vice versa.

11.1 The model

Let us start by taking the model of Sec. 1.5.3 and then take the strong gauge coupling limit $e \to \infty, g \to \infty$, and set the masses to zero $m = 0$ but keeping $\kappa \neq \mu$. Physically, this sends the masses of the gauge bosons associated with the Yang-Mills-Higgs mechanism to infinity and we can thus integrate out the adjoint scalar field ϕ:

$$\phi^a = \frac{4\pi}{\mu} \mathrm{Tr}\left(HH^\dagger t^a\right) , \quad \phi^0 = \frac{4\pi}{\kappa}\frac{1}{\sqrt{2N}}\left[\mathrm{Tr}\left(HH^\dagger\right) - \xi\right] . \tag{11.1}$$

This leaves us with the non-Abelian Chern-Simons theory

$$\begin{aligned}\mathcal{L}_{\mathrm{CSH}} = &-\frac{\mu}{8\pi}\epsilon^{\mu\nu\rho}\left(A^a_\mu\partial_\nu A^a_\rho - \frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right) - \frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}\left(A^0_\mu\partial_\nu A^0_\rho\right) + \mathrm{Tr}\left(\mathcal{D}_\mu H\right)^\dagger\left(\mathcal{D}^\mu H\right)\\ &- 4\pi^2\mathrm{Tr}\left|\left\{\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right) + \frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right\}H\right|^2 ,\end{aligned} \tag{11.2}$$

which will be the focus of this Chapter. It still enjoys $\mathcal{N} = 2$ supersymmetry and there are 3 parameters governing the solutions; the Abelian Chern-Simons coupling κ and the non-Abelian Chern-Simons coupling μ and finally the winding number $\nu = \frac{k}{n_0}$ [6]. n_0 denotes the greatest common divisor (gcd) of the Abelian charges of the holomorphic invariants of G', see [6]. For simple groups this coincides with the center as $\mathbb{Z}_{n_0}$. We will take $k > 0$.

There are three different phases of the theory at hand. An unbroken phase with $\langle H \rangle = 0$ and a broken phase with $\langle H \rangle = \sqrt{\frac{\xi}{N}}$. In between there are partially broken phases. We will consider only the completely broken phase in this paper.

The equations of motion are

$$\frac{\mu}{8\pi}\epsilon^{\mu\nu\sigma}F^a_{\mu\nu} = -i\mathrm{Tr}\left[H^\dagger t^a \mathcal{D}^\sigma H - \left(\mathcal{D}^\sigma H\right)^\dagger t^a H\right] \,, \tag{11.3}$$

$$\frac{\kappa}{8\pi}\epsilon^{\mu\nu\sigma}F^0_{\mu\nu} = -i\mathrm{Tr}\left[H^\dagger t^0 \mathcal{D}^\sigma H - \left(\mathcal{D}^\sigma H\right)^\dagger t^0 H\right] \,, \tag{11.4}$$

$$\begin{aligned}\mathcal{D}_\mu\mathcal{D}^\mu H = &-4\pi^2\left[\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right]^2 H \\ &-\frac{8\pi^2}{N\kappa}\mathrm{Tr}\left(\left[\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right]HH^\dagger\right)H \\ &-\frac{16\pi^2}{\mu}\mathrm{Tr}\left(\left[\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^b\right)t^b\right]HH^\dagger t^a\right)t^a H\,.\end{aligned} \tag{11.5}$$

The tension, defined by the integral on the plane over the time-time component of the energy-momentum tensor, is given by

$$T = \int_{\mathbb{C}}\mathrm{Tr}\left\{\left|\mathcal{D}_0 H\right|^2+\left|\mathcal{D}_i H\right|^2+4\pi^2\left|\left(\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right)H\right|^2\right\}, \tag{11.6}$$

which by a standard Bogomol'nyi completion can be rewritten as

$$\begin{aligned}T = &\int_{\mathbb{C}}\mathrm{Tr}\left\{\left|\mathcal{D}_0 H - i2\pi\left(\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right)H\right|^2+4\left|\bar{\mathcal{D}}H\right|^2\right\} \\ &-\frac{\xi}{\sqrt{2N}}\int_{\mathbb{C}}F^0_{12}+i\mathrm{Tr}\int_{\mathbb{C}}\left[\partial_2\left(H^\dagger\mathcal{D}_1 H\right)-\partial_1\left(H^\dagger\mathcal{D}_2 H\right)\right]\,.\end{aligned} \tag{11.7}$$

This leads immediately to the BPS-equations which need to be accompanied by the Gauss law being the $\sigma = 0$ component of the Eqs. (11.3),(11.4)

$$\bar{\mathcal{D}}H = 0\,, \quad \mathcal{D}_0 H = i2\pi\left(\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right)H\,. \tag{11.8}$$

Rewriting the boundary term using the first BPS-equation, we have for the BPS saturated vortices the tension

$$T = -\frac{\xi}{\sqrt{2N}}\int_{\mathbb{C}}F^0_{12}+\frac{1}{2}\mathrm{Tr}\int_{\mathbb{C}}\partial_i^2\left(HH^\dagger\right) = 2\pi\xi\nu\,, \tag{11.9}$$

with ν being the $U(1)$ winding number. By combining the BPS equations with the Gauss law, we obtain the following system

$$\bar{\mathcal{D}}H = 0\ , \tag{11.10}$$

$$F_{12}^a t^a = \frac{16\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right)\mathrm{Tr}\left(HH^\dagger t^a\right)t^a + \frac{16\pi^2}{\mu^2}\mathrm{Tr}\left(HH^\dagger t^b\right)\mathrm{Tr}\left(HH^\dagger\left\{t^a, t^b\right\}\right)t^a\ , \tag{11.11}$$

$$F_{12}^0 t^0 = \frac{8\pi^2}{N^2\kappa^2}\mathrm{Tr}\left(HH^\dagger\right)\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right)\mathbf{1}_N + \frac{16\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\left(HH^\dagger t^a\right)\right)^2\mathbf{1}_N\ . \tag{11.12}$$

An interesting comment is that the system only depends on three combinations of the couplings; viz. κ^2, μ^2 and $\kappa\mu$. There are thus only two choices of signs giving different solutions $\mathrm{sign}(\kappa) = \pm\mathrm{sign}(\mu)$. This system is of a generic character and one can readily apply one's favorite group. Setting $\kappa = \mu$, the BPS-equations become

$$\bar{\mathcal{D}}H = 0\ , \quad \mathcal{D}_0 H = \frac{i2\pi}{\kappa}\left[2\mathrm{Tr}\left(HH^\dagger t^\alpha\right)t^\alpha - \frac{\xi}{N}\mathbf{1}_N\right]H\ , \tag{11.13}$$

which in turn yield the simplified system by combination with the Gauss law

$$\bar{\mathcal{D}}H = 0\ , \quad F_{12}^\alpha t^\alpha = \frac{16\pi^2}{\kappa^2}\left[\mathrm{Tr}\left(HH^\dagger\left\{t^\alpha, t^\beta\right\}\right)\mathrm{Tr}\left(HH^\dagger t^\beta\right) - \frac{\xi}{N}\mathrm{Tr}\left(HH^\dagger t^\alpha\right)\right]t^\alpha\ . \tag{11.14}$$

In the next Section, we will consider the cases of $G' = SU(N)$, $G' = SO(N)$ and $G' = USp(2M)$, and finally make the corresponding master equations.

11.2 Master equations

11.2.1 $G = U(1) \times SU(N)$

Considering the case of $U(1) \times SU(N)$, the BPS-equations combined with the Gauss law read

$$\bar{\mathcal{D}}H = 0\ , \tag{11.15}$$

$$\begin{aligned} F_{12}^a t^a &= \frac{8\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right)\left(HH^\dagger - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(HH^\dagger\right)\right) \\ &\quad + \frac{8\pi^2}{\mu^2}\Bigg[HH^\dagger\left(HH^\dagger - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(HH^\dagger\right)\right) \\ &\qquad - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\left(HH^\dagger\right)^2\right) + \frac{\mathbf{1}_N}{N^2}\left(\mathrm{Tr}\left(HH^\dagger\right)\right)^2\Bigg]\ , \end{aligned} \tag{11.16}$$

$$\begin{aligned} F_{12}^0 t^0 &= \frac{8\pi^2}{N^2\kappa^2}\mathrm{Tr}\left(HH^\dagger\right)\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right)\mathbf{1}_N \\ &\quad + \frac{8\pi^2}{N\kappa\mu}\left[\mathrm{Tr}\left(\left(HH^\dagger\right)^2\right) - \frac{1}{N}\left(\mathrm{Tr}\left(HH^\dagger\right)\right)^2\right]\mathbf{1}_N\ . \end{aligned}$$

In this case, the generic vacuum is given by

$$\langle H \rangle = \sqrt{\frac{\xi}{N}}\mathbf{1}_N \,. \tag{11.17}$$

This vacuum allows for an unbroken global symmetry, the so-called color-flavor symmetry which is the global part of the gauge transformation combined with the flavor symmetry. This is of crucial importance for having orientational modes in vortex configurations.

Utilizing the moduli matrix formalism, we can immediately solve the first BPS-equation and rewrite the second in terms of the new variables

$$H = S^{-1}H_0(z) \,, \quad \bar{A}^a t^a = -iS'^{-1}\bar{\partial}S' \,, \quad \bar{A}^0 t^0 = -i\bar{\partial}\log s \tag{11.18}$$

along with the definitions $\Omega \equiv \omega\Omega', \Omega' \equiv S'S'^{\dagger}, \omega \equiv ss^{\dagger}$ and $\Omega_0 \equiv H_0(z)H_0^{\dagger}(z)$. The field-strength matrices are

$$F_{12}^a t^a = 2S'^{-1}\bar{\partial}\left[\Omega'\partial\Omega'^{-1}\right]S' \,, \qquad F_{12}^0 t^0 = -2\mathbf{1}_N\bar{\partial}\partial\log\omega \,. \tag{11.19}$$

In this $U(1) \times SU(N)$ case we can write down the two master equations like

$$\begin{aligned}
\bar{\partial}\left[\Omega'\partial\Omega'^{-1}\right] = \frac{4\pi^2}{N\kappa\mu}\frac{1}{\omega}\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right) - \xi\right)\left(\Omega_0\Omega'^{-1} - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right) \\
+ \frac{4\pi^2}{\mu^2}\frac{1}{\omega^2}\Bigg[\Omega_0\Omega'^{-1}\left(\Omega_0\Omega'^{-1} - \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right) \\
- \frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\left(\Omega_0\Omega'^{-1}\right)^2\right) + \frac{\mathbf{1}_N}{N^2}\left(\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)^2\Bigg] \,,
\end{aligned} \tag{11.20}$$

$$\begin{aligned}
\bar{\partial}\partial\log\omega = -\frac{4\pi^2}{N^2\kappa^2}\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right) - \xi\right) \\
+ \frac{4\pi^2}{N\kappa\mu}\frac{1}{\omega^2}\left[\mathrm{Tr}\left(\left(\Omega_0\Omega'^{-1}\right)^2\right) - \frac{1}{N}\left(\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)^2\right] \,.
\end{aligned} \tag{11.21}$$

Setting the couplings equal $\kappa = \mu$, we can write the $U(N)$ Chern-Simons BPS-equations and master equation as simple as

$$F_{12}^\alpha t^\alpha = \frac{8\pi^2}{\kappa^2}HH^\dagger\left(HH^\dagger - \frac{\xi}{N}\mathbf{1}_N\right) \,, \tag{11.22}$$

$$\bar{\partial}\left[\Omega\partial\Omega^{-1}\right] = \frac{4\pi^2}{\kappa^2}\Omega_0\Omega^{-1}\left[\Omega_0\Omega^{-1} - \frac{\xi}{N}\mathbf{1}_N\right] \,. \tag{11.23}$$

The boundary conditions for these master equations coincide with the weak coupling solutions (11.48).

11.2.2 $G = U(1) \times SO(N)$ and $G = U(1) \times USp(2M)$

Considering now the gauge groups $G = U(1) \times SO(N)$ and $G = U(1) \times USp(2M)$ on the same footing with their corresponding invariant tensor J, which has the properties $J^\dagger J = \mathbf{1}_N$ and $J^{\mathrm{T}} = \epsilon J$ with $\epsilon = \pm 1$ for $SO(N)$ and $USp(2M)$, respectively.

The vacuum has the generic form [5]

$$\langle H \rangle = \mathrm{diag}\,(v_1, v_2, \ldots, v_N) \ , \quad v_i \in \mathbb{R}_+ \ , \tag{11.24}$$

however, we will consider the most symmetric vacuum allowing for the global color-flavor symmetry, viz. we will here use (11.17). We have the following system which is obtained by combining the BPS equations with the Gauss law and applying respective algebras

$$\begin{aligned}
\bar{\mathcal{D}}H &= 0 \ , \\
F_{12}^a t^a &= \frac{4\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right)\left(HH^\dagger - J^\dagger\left(HH^\dagger\right)^{\mathrm{T}} J\right) \\
&\quad + \frac{2\pi^2}{\mu^2}\left[\left(HH^\dagger\right)^2 - J^\dagger\left(\left(HH^\dagger\right)^2\right)^{\mathrm{T}} J\right] \ , \\
F_{12}^0 t^0 &= \frac{8\pi^2}{N^2\kappa^2}\mathrm{Tr}\left(HH^\dagger\right)\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right)\mathbf{1}_N \\
&\quad + \frac{4\pi^2}{N\kappa\mu}\mathrm{Tr}\left(HH^\dagger\left(HH^\dagger - J^\dagger\left(HH^\dagger\right)^{\mathrm{T}} J\right)\right)\mathbf{1}_N \ ,
\end{aligned} \tag{11.25}$$

which lead to the master equations

$$\begin{aligned}
\bar{\partial}\left[\Omega'\partial\Omega'^{-1}\right] &= \frac{2\pi^2}{N\kappa\mu}\frac{1}{\omega}\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right) - \xi\right)\left(\Omega_0\Omega'^{-1} - J^\dagger\left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}} J\right) \\
&\quad + \frac{\pi^2}{\mu^2}\frac{1}{\omega^2}\left[\left(\Omega_0\Omega'^{-1}\right)^2 - J^\dagger\left(\left(\Omega_0\Omega'^{-1}\right)^2\right)^{\mathrm{T}} J\right] \ ,
\end{aligned} \tag{11.26}$$

$$\begin{aligned}
\bar{\partial}\partial\log\omega &= -\frac{4\pi^2}{N^2\kappa^2}\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right) - \xi\right) \\
&\quad - \frac{2\pi^2}{N\kappa\mu}\frac{1}{\omega^2}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\left(\Omega_0\Omega'^{-1} - J^\dagger\left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}} J\right)\right) \ .
\end{aligned} \tag{11.27}$$

The boundary conditions for these master equations coincide with the weak coupling solutions (11.52).

11.2.3 Energy density and flux densities

Rewriting the energy density (11.9) in terms of our new variables and remembering the boundary term which vanishes when integrating over the entire plane, while it nevertheless produces a big difference between the magnetic flux density and the energy density, we have

$$\mathcal{E} = 2\xi\bar{\partial}\partial\log\omega + 2\bar{\partial}\partial\left(\frac{1}{\omega}\mathrm{Tr}\Omega_0\Omega'^{-1}\right) \ . \tag{11.28}$$

However, the total energy

$$E = \int_{\mathbb{C}} \mathcal{E} = 2\pi\xi\nu = \frac{2\pi\xi k}{n_0} \ , \tag{11.29}$$

is simply proportional to the topological charge as always.

The Abelian magnetic flux density is the first term (up to a factor) in the energy density

$$\mathcal{B} = F_{12}^0 = -2\sqrt{2N}\;\bar{\partial}\partial\log\omega \;, \tag{11.30}$$

whereas the non-Abelian flux is the matrix defined in Eq. (11.19). The Abelian electric field density reads

$$E_i = F_{i0}^0 = \frac{2\pi}{\kappa}\sqrt{\frac{2}{N}}\partial_i\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right) \;, \tag{11.31}$$

while the non-Abelian electric field density is given by

$$E_i^a t^a = F_{i0}^a t^a = \frac{4\pi}{\mu}\partial_i\mathrm{Tr}\left(HH^\dagger t^a\right)t^a \;. \tag{11.32}$$

This can be written for $G' = SU(N)$ as

$$E_i^a t^a = \frac{2\pi}{\mu}\partial_i\left[\frac{1}{\omega}\left(S'^{-1}\Omega_0\Omega'^{-1}S' - \frac{1}{N}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)\right] \;, \tag{11.33}$$

while for $G' = SO(N)$ or $G' = USp(2M)$ it is

$$E_i^a t^a = \frac{\pi}{\mu}\partial_i\left[\frac{1}{\omega}S'^{-1}\left(\Omega_0\Omega'^{-1} - J^\dagger\left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}}J\right)S'\right] \;. \tag{11.34}$$

11.3 Solutions

In the Abelian Chern-Simons theory, there exists a rigorous existence proof of the solutions in Ref. [258]. To our knowledge this has not rigorously been proved in the theory at hand. In the case of the vortices in the Yang-Mills-Higgs theory, the "covariant holomorphic" condition on the Higgs fields $\bar{\mathcal{D}}H = 0$, which is solved by the moduli matrix formalism, does uniquely determine the full moduli space of vortices via the Hitchin-Kobayashi correspondence [259, 260, 261, 134], which however has only been proved on compact spaces. This means that the corresponding master equations do not induce further moduli. For the vortices with the $U(N)$ gauge group, an index theorem has been given in Ref. [9] while for generic gauge groups (under certain conditions) an index theorem has been given in Ref. [3]. The index computed gives the number of moduli and does indeed correspond to the number of moduli found in the moduli matrix.

The first part of constructing a solution is to write down the moduli matrix. Here we simply follow the way paved by the paper [6] using holomorphic invariants of the gauge subgroup G'. This boils down to some constraints for the moduli matrix to obey. A few examples of interest here is the case of $G' = SU(N)$

$$\det H_0(z) = z^k + \mathcal{O}\left(z^{k-1}\right) \;, \tag{11.35}$$

while in the case of $G' = SO, USp$, respectively, we have

$$H_0^{\mathrm{T}}(z)JH_0(z) = z^{\frac{2k}{n_0}}J + \mathcal{O}\left(z^{\frac{2k}{n_0}-1}\right) \;, \tag{11.36}$$

where k is the vortex number (recall that $\nu = \frac{k}{n_0}$ is the $U(1)$ winding) and $n_0 = 2$ in case of $SO(2M)$ and $USp(2M)$ while $n_0 = 1$ for $SO(2M+1)$, M being positive integers. For $SU(N)$, however $n_0 = N$.

The rather complicated looking master equations found in the last Section are assumed to have a unique solution for each moduli matrix $H_0(z)$ (up to V equivalence, see Ref. [6]). That is the moduli matrices are redundant and have to be identified by the following V transformation

$$H_0(z) \sim V(z,\bar{z})H_0(z) \ , \quad S(z,\bar{z}) \sim V(z,\bar{z})S(z,\bar{z}) \ , \quad V \in G^{\mathbb{C}} \ . \tag{11.37}$$

Here we conjecture the existence and uniqueness of the solutions to the master equation for each moduli matrix (up to the V equivalence). To provide plausibility for this claim we shall continue in two directions.

First we consider the weak coupling limit $\kappa \to 0$ and $\mu \to 0$, which seems like an odd limit to take, but having an advantage. Looking at the theory (1.202) it is immediately seen that the matter fields are forced to stay in the vacuum manifold corresponding to the *strong* coupling limit of the normal non-Abelian vortex (i.e. with only a Yang-Mills kinetic term). In turn, this gives us a unique solution which in fact is the same solution as found in the strong coupling limit of the non-Abelian vortex with only a Yang-Mills kinetic term. This solution, appropriate only for vortices of the semi-local type, are usually called lumps in the literature.

The second direction we will take will simply be to find some solutions by numerical calculations.

Now the existence of the solutions to the master equations, as we argue, makes it possible to exploit a lot of results developed in the literature. In short,

the moduli space of non-Abelian Chern-Simons k vortices with gauge group G is equal to the moduli space of the non-Abelian Yang-Mills k vortices with gauge group G. (11.38)

Moduli spaces of the non-Abelian vortices in $\mathcal{N} = 2$ SQCD has been found in the literature in Refs. [9, 100] for $U(N)$ and in Refs. [3] for $SO(N), USp(2M)$.

Here we will summarize a few results from the literature. In the pioneering papers [9, 97] discovering the non-Abelian vortices with gauge group $U(N)$ (in contrast to the formerly found $\mathbb{Z}_N$ strings) the moduli space of a single vortex string was found to be

$$\mathcal{M}_{k=1,G'=SU(N)} = \mathbb{C} \times \mathbb{C}P^{N-1}, \tag{11.39}$$

where the first factor denotes the position in the transverse plane while the second factor are orientational modes. For well separated k vortices, the moduli space can be composed as simply the symmetric product of that of the single vortex. This is not the case, when the centers coincide. In the $k = 2$, $U(2)$ case, the moduli space has been found explicitly in the Refs. [110, 111]

$$\mathcal{M}_{k=2,G'=U(2)} = \mathbb{C} \times W\mathbb{C}P^2_{2,1,1} \ , \tag{11.40}$$

which decomposes into a center-of-mass position and a weighted complex projective space with unequal weights giving rise to a conical type of singularity. In Ref. [3] the moduli spaces of vortices with gauge groups $G = U(1) \times SO(N)$ and $G = U(1) \times USp(2M)$ have been found. A complication arises due to the fact that already for $N_\mathrm{F} = N$ flavors, the vortices are in general of the semi-local type (i.e. they have polynomial tails in their profile functions). The spaces quoted here correspond to the vortices of local type, thus they are constrained to have holomorphic invariants with coincident zeroes. In the language of Ref. [3] this is obtained by constraining the vortices by the so-called strong condition

$$H_0^\mathrm{T}(z) J H_0(z) = (z - z_0)^{\frac{2k}{n_0}} J \; . \tag{11.41}$$

The single local vortex with $G' = USp(2M)$ has the moduli space

$$\mathcal{M}_{k=1,G'=USp(2M)} = \mathbb{C} \times \frac{USp(2M)}{U(M)} \; , \tag{11.42}$$

while in the case of $G' = SO(2M)$ it is found to be

$$\mathcal{M}_{k=1,G'=SO(2M)} = \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_+ \cup \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_- \; , \tag{11.43}$$

where the $\pm$ denotes the chirality as described in detail in Ref. [3] which is deeply rooted in the fact that the first homotopy group has in addition to the integers a $\mathbb{Z}_2$ factor. This can also be interpreted as two spinor representations which is exactly the irreducible representations of the dual group $\tilde{G}'$, where the dual is defined as being the group having the root vectors $\vec{\alpha}^* = \frac{\vec{\alpha}}{\vec{\alpha}\cdot\vec{\alpha}}$. For the $k = 2$, $G' = SO(2M)$ the following orientational moduli spaces have been found to be locally

$$\mathcal{M}_{k=2,G'=SO(4m),Q_{\mathbb{Z}_2}=+1} = \mathbb{R}_+^m \times \frac{SO(4m)}{USp(2)^m} \times \mathbb{Z}_2 \; , \tag{11.44}$$

$$\mathcal{M}_{k=2,G'=SO(4m),Q_{\mathbb{Z}_2}=-1} = \mathbb{R}_+^{m-1} \times \frac{SO(4m)}{U(1) \times USp(2)^{m-1} \times SO(2)} \; , \tag{11.45}$$

$$\mathcal{M}_{k=2,G'=SO(4m+2),Q_{\mathbb{Z}_2}=+1} = \mathbb{R}_+^m \times \frac{SO(4m+2)}{U(1) \times USp(2)^m} \times \mathbb{Z}_2 \; , \tag{11.46}$$

$$\mathcal{M}_{k=2,G'=SO(4m+2),Q_{\mathbb{Z}_2}=-1} = \mathbb{R}_+^m \times \frac{SO(4m+2)}{USp(2)^m \times SO(2)} \; . \tag{11.47}$$

In the case of $k = 1$, $G' = SO(2M + 1)$, the moduli spaces are quite similar to the $k = 2$ even case.

11.4 Weak coupling limit

11.4.1 $G = U(1) \times SU(N)$

Taking $\kappa = \mu \to 0$, we obtain from the D term conditions

$$\Omega' = (\det \Omega_0)^{-\frac{1}{N}} \, \Omega_0 \; , \quad \omega = \frac{N}{\xi} (\det \Omega_0)^{\frac{1}{N}} \; , \quad \Omega = \frac{N}{\xi} \Omega_0 \; , \tag{11.48}$$

which can be packaged together as a $U(N)$ field Ω. Instead of taking both couplings simultaneously to weak coupling, we can play a game of taking only one of them, keeping the other finite (non-infinitesimal). Taking $\kappa \to 0$ and keeping μ finite we obtain

$$\omega = \frac{1}{\xi}\mathrm{Tr}\Omega_0\Omega'^{-1} , \tag{11.49}$$

at the zeroth order in κ while at first order we get the constraint

$$N\mathrm{Tr}\left(\left(\Omega_0\Omega'^{-1}\right)^2\right) = \left(\mathrm{Tr}\,\Omega_0\Omega'^{-1}\right)^2 . \tag{11.50}$$

We note that only the Abelian field is determined, however at first order in the coupling constant we obtain a single constraint on the non-Abelian fields. Taking instead $\mu \to 0$ keeping κ finite we have

$$\Omega' = \Lambda\Omega_0 , \qquad \text{with } \Lambda \in \text{const.} , \tag{11.51}$$

to both zeroth and first order in μ.

11.4.2 $G = U(1) \times SO(N)$ and $G = U(1) \times USp(2M)$

Taking $\kappa = \mu \to 0$ we have from the D term conditions [3, 5]

$$\Omega' = H_0(z)\frac{\mathbf{1}_N}{\sqrt{M^\dagger M}}H_0^\dagger(z) , \qquad \omega = \frac{1}{\xi}\mathrm{Tr}\sqrt{M^\dagger M} , \tag{11.52}$$

where $M = H_0^{\mathrm{T}}(z)JH_0(z)$ is the meson field of the SO, USp theories according to the choice of the gauge group and in turn invariant tensor.

A comment in store is that the Chern-Simons term is simply switched off in this limit and the lumps are *the same* as the ones living in the Yang-Mills theories experiencing infinitely massive gauge bosons. The point here, however, is to argue by continuity the existence and uniqueness of the solutions to the master equations for a given moduli matrix $H_0(z)$ (up to the V-equivalence relation).

11.5 Numerical solutions

11.5.1 Example: $U(N)$

Let us do a warm-up and consider the single $U(N)$ Chern-Simons vortex ($\kappa = \mu$) as has been found in Refs. [144, 143], however doing it in our formalism. Taking a simple moduli matrix

$$H_0(z) = \mathrm{diag}\left(z, \mathbf{1}_{N-1}\right) , \tag{11.53}$$

which of course satisfies the constraint (11.35), thus we can use the Ansatz for Ω

$$\Omega = e^{\psi}\mathrm{diag}\left(e^{(N-1)\chi}, e^{-\chi}\mathbf{1}_{N-1}\right) , \tag{11.54}$$

leading to the two coupled equations of motion

$$\bar{\partial}\partial\left[\psi+(N-1)\chi\right]=-\frac{4\pi^2}{\kappa^2}\left|z-z_0\right|^2 e^{-\psi-(N-1)\chi}\left(\left|z-z_0\right|^2 e^{-\psi-(N-1)\chi}-\frac{\xi}{N}\right)\,, \tag{11.55}$$

$$\bar{\partial}\partial\left[\psi-\chi\right]=-\frac{4\pi^2}{\kappa^2}e^{-\psi+\chi}\left(e^{-\psi+\chi}-\frac{\xi}{N}\right)\,. \tag{11.56}$$

Notice that the two equations decouple in the sense that there only appear the combinations $\psi+(N-1)\chi$ and $\psi-\chi$. In fact it is easily seen that in this case, the field combination $\psi-\chi$ can be in the vacuum in all $\mathbb{C}$ which trivially solves the second equation. However, the first equation still needs to be solved numerically. The boundary conditions are

$$\psi_\infty=\log\left(\frac{N|z|^{\frac{2}{N}}}{\xi}\right)\,,\qquad \chi_\infty=\log\left(|z|^{\frac{2}{N}}\right)\,. \tag{11.57}$$

The equations become essentially Abelian when the couplings are equal $\kappa=\mu$, as was noted in Ref. [144]. The energy density is given by

$$\mathcal{E}=2\xi\bar{\partial}\partial\psi+2\bar{\partial}\partial\left[|z|^2e^{-\psi-(N-1)\chi}+(N-1)e^{-\psi+\chi}\right]\,, \tag{11.58}$$

where the last term is the boundary term which of course integrates to zero. The Abelian and non-Abelian magnetic flux densities are given by

$$F_{12}^0=-2\sqrt{2N}\;\bar{\partial}\partial\psi\,,\quad F_{12}^a t^a=-2\sqrt{2N(N-1)}\;\bar{\partial}\partial\chi\, t\,, \tag{11.59}$$

where the following matrix has been defined for convenience

$$t\equiv\frac{1}{\sqrt{2N(N-1)}}\mathrm{diag}\left(N-1,-\mathbf{1}_{N-1}\right)\,, \tag{11.60}$$

which is traceless and has the trace of its square normalized to one half. The Abelian electric field density is

$$E_r=\frac{2\pi}{\kappa}\sqrt{\frac{2}{N}}\partial_r\left[r^2e^{-\psi-(N-1)\chi}+(N-1)e^{-\psi+\chi}\right]\,, \tag{11.61}$$

while the non-Abelian electric field density is

$$E_r^a t^a=\frac{2\pi}{\mu N}\sqrt{2N(N-1)}\partial_r\left[r^2e^{-\psi-(N-1)\chi}-e^{-\psi+\chi}\right]t\,. \tag{11.62}$$

We will find in the next Subsection, that the numerical solution for this vortex for $N=2$ is up to rescaling of some parameters equivalent to the vortex studied in the next Subsection (when $\kappa=\mu$). Thus the concrete graphs are shown only for the vortex solution below.

11.5.2 Example: $U(1) \times SO(2M)$ and $U(1) \times USp(2M)$

Let us take a simple example of a moduli matrix

$$H_0(z) = \text{diag}\,(z\mathbf{1}_M, \mathbf{1}_M) \ , \tag{11.63}$$

which surely satisfies the constraint (11.36). We take the Ansatz

$$\Omega' = \text{diag}\left(e^{\chi}\mathbf{1}_M, e^{-\chi}\mathbf{1}_M\right) \ , \quad \omega = e^{\psi} \ , \tag{11.64}$$

where $\det \Omega' = 1$ is manifest. The equations of motion in terms of the new fields are

$$\begin{aligned}
\bar{\partial}\partial\chi = & -\frac{\pi^2}{\kappa\mu}\left(|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi} - \frac{\xi}{M}\right)\left(|z|^2 e^{-\psi-\chi} - e^{-\psi+\chi}\right) \\
& -\frac{\pi^2}{\mu^2}\left(\left(|z|^2 e^{-\psi-\chi}\right)^2 - \left(e^{-\psi+\chi}\right)^2\right) \ ,
\end{aligned} \tag{11.65}$$

$$\begin{aligned}
\bar{\partial}\partial\psi = & -\frac{\pi^2}{\kappa^2}\left(|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi}\right)\left(|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi} - \frac{\xi}{M}\right) \\
& -\frac{\pi^2}{\kappa\mu}\left(|z|^2 e^{-\psi-\chi} - e^{-\psi+\chi}\right)^2 \ .
\end{aligned} \tag{11.66}$$

It is interesting to note that under rescaling of the FI parameter $\xi \to M\xi$, the above equations of motion are exactly the ones of the $U(1) \times SU(2)$ theory with the Ansatz used in the last Section. The boundary conditions are

$$\psi_\infty = \log\left(\frac{2M}{\xi}|z|\right) \ , \quad \chi_\infty = \log\left(|z|\right) \ , \tag{11.67}$$

and the energy density reads

$$\mathcal{E} = 2\xi\bar{\partial}\partial\psi + 2M\bar{\partial}\partial\left[|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi}\right] \ , \tag{11.68}$$

where the first term is proportional to the Abelian magnetic flux density

$$F_{12}^0 = -4\sqrt{M}\ \bar{\partial}\partial\psi \ , \tag{11.69}$$

and the last is the boundary term which integrates to zero, while the non-Abelian magnetic field density reads

$$F_{12}^a t^a \equiv F_{12}^{\text{NA}} t = -4\sqrt{M}\ \bar{\partial}\partial\chi\, t \ , \quad t \equiv \frac{1}{2\sqrt{M}}\text{diag}\,(\mathbf{1}_M, -\mathbf{1}_M) \ . \tag{11.70}$$

The Abelian electric field density reads

$$E_r = \frac{2\pi\sqrt{M}}{\kappa}\partial_r\left[r^2 e^{-\psi-\chi} + e^{-\psi+\chi}\right] \ , \tag{11.71}$$

whereas the non-Abelian electric field density is

$$E_r^a t^a \equiv E_r^{\text{NA}} t = \frac{2\pi\sqrt{M}}{\mu}\partial_r\left[r^2 e^{-\psi-\chi} - e^{-\psi+\chi}\right] t \ . \tag{11.72}$$

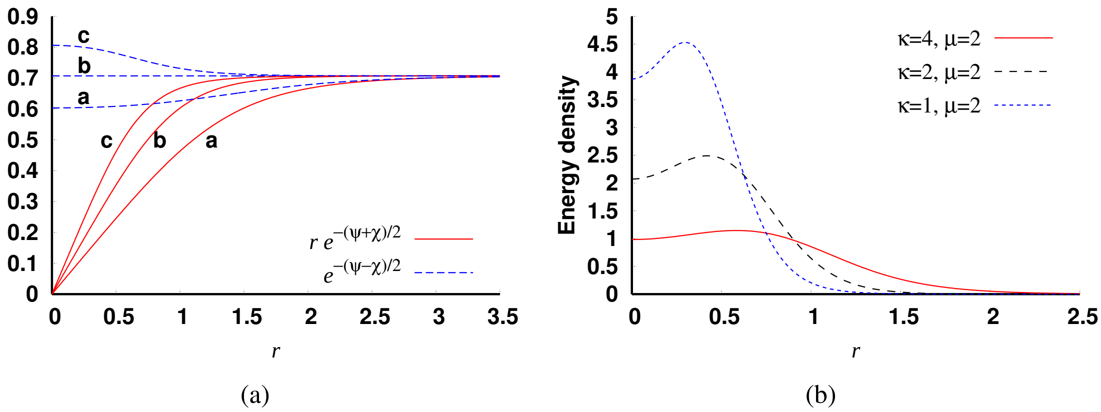


Figure 11.1: (a) Profile functions for three different values of the coupling constants; a: $\kappa = 4, \mu = 2$; b: $\kappa = 2, \mu = 2$; c: $\kappa = 1, \mu = 2$; the functions are plotted in traditional style with the winding field rising linearly and the non-winding field being constant at the origin. The FI parameter $\xi = 2$. Notice that the VEV for these functions is $2^{-\frac{1}{2}}$. (b) The energy density $\mathcal{E}$ for the vortex for the same three different values of the couplings. All the energy densities integrate to $\pi\xi$, within an accuracy better than $\sim 10^{-4}$.

We show the vortex with this Ansatz corresponding to different values of the coupling constants κ, μ in the following figures. Here we will take for definiteness the group G' to be $SO(4)$ or $USp(4)$ hence $M = 2$, which within the chosen Ansatz are equivalent. We furthermore set $\xi = 2$. The total energy is thus (recall the Ansatz is for a single $k = 1$ vortex)

$$E = \int_{\mathbb{C}} \mathcal{E} = \pi\xi \; . \tag{11.73}$$

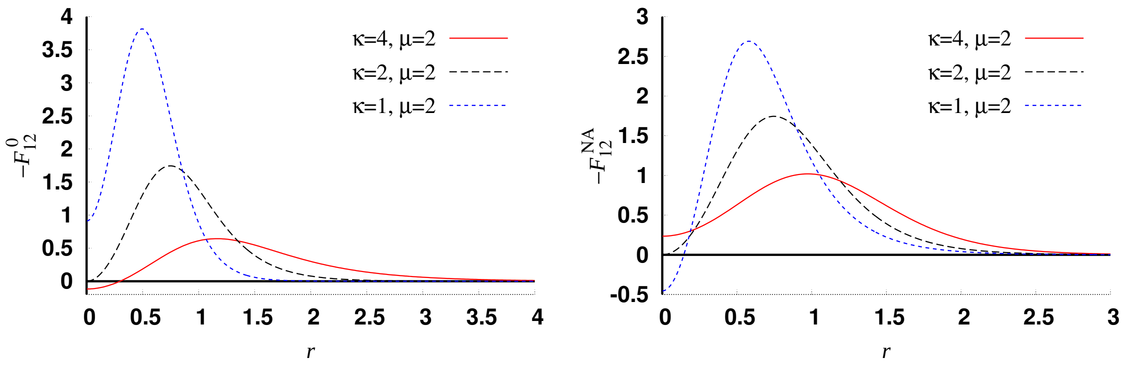
In Fig. 11.1a we show the profile functions of the vortex in the traditional way, where the color-flavor matrix is parametrized as follows

$$H = \text{diag}\left(f(r)e^{i\theta}\mathbf{1}_2, g(r)\mathbf{1}_2\right) \;, \tag{11.74}$$

which of course is equivalent to the parametrization in terms of ψ, χ. In Fig. 11.1b the energy density of Eq. (11.68) is shown. The integral of the energy density is identically equal to the integral of the Abelian magnetic flux, as it should be. We see the vortex size is proportional to the coupling constants. In Fig. 11.2 we show the Abelian (a) and the non-Abelian (b) magnetic field, respectively. We observe that the Abelian magnetic field is negative at the origin while the non-Abelian magnetic field is positive, in the $\kappa = 4, \mu = 2$ case. The contrary holds in the $\kappa = 1, \mu = 2$ case where the non-Abelian magnetic field is negative at the origin while the Abelian field is positive. It turns out that the combination

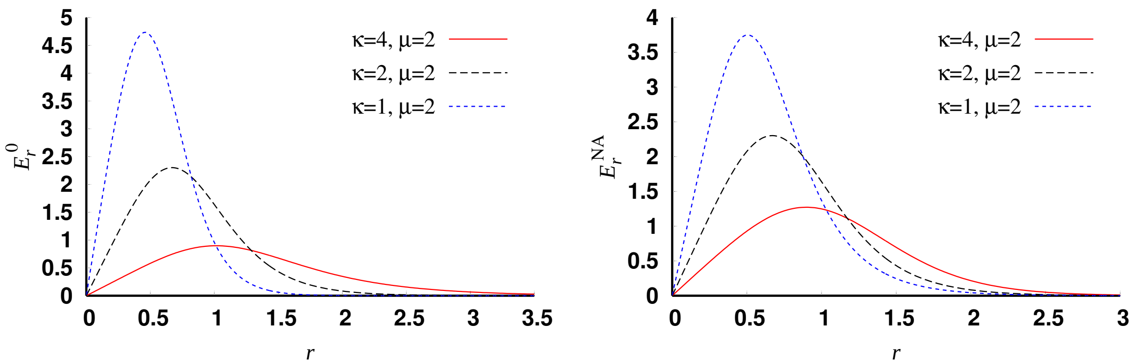
$$\left(\kappa F_{12}^{0} + \mu F_{12}^{\text{NA}}\right)\Big|_{r\to 0} = 0 \; . \tag{11.75}$$

An immediate consequence is that for $|\kappa| \gg |\mu|$, $|F_{12}^{\text{NA}}| \gg |F_{12}^{0}|$ at the origin and vice versa. Plots of the Abelian and non-Abelian magnetic fields normalized as in Eq. (11.75) are shown in Fig. 11.4 with $\kappa = 4, \mu = 2$ in (a) and $\kappa = 1, \mu = 2$ in (b), respectively. At



(a) (b)

Figure 11.2: (a) The Abelian magnetic field F_{12}^0 (trace-part) for three different values of the couplings. Notice the equal coupling case has zero magnetic field at the origin while the different coupling cases have negative and positive values, respectively. (b) The non-Abelian magnetic field F_{12}^a (traceless part) for different values of the couplings. Notice the opposite behavior of the non-Abelian magnetic field with respect the Abelian one at the origin, see also Fig. 11.4. The FI parameter $\xi = 2$.



(a) (b)

Figure 11.3: (a) The Abelian electric field in the radial direction E_r (trace-part) for three different values of the couplings. (b) The non-Abelian electric field in the radial direction E_r^{NA} (traceless part). The FI parameter $\xi = 2$.

the origin this combination cancels to a numerical accuracy better than 10^{-5}. First let us demonstrate the formula (11.75) by calculating the fields in the limit $r \to 0$

$$\kappa F^0_{12}\big|_{r\to 0} = -\mu F^{\mathrm{NA}}_{12}\big|_{r\to 0} = 4\pi\sqrt{M}\left[\frac{1}{\kappa}e^{-\psi+\chi}\left(e^{-\psi+\chi} - \frac{\xi}{M}\right) + \frac{1}{\mu}\left(e^{-\psi+\chi}\right)^2\right] . \quad (11.76)$$

Note that the value of the magnetic fields only depends on the field combination $\psi - \chi$, and it is understood that it has to be evaluated at the origin in the above equation. Secondly, let us demonstrate that the magnetic fields are zero at the origin in the case of equal couplings. Subtracting Eq. (11.65) from Eq. (11.66) we have

$$\begin{aligned}\bar{\partial}\partial\left(\psi - \chi\right) = -\frac{\pi^2}{\kappa^2}\bigg[&\left(1 - \frac{\kappa^2}{\mu^2}\right)\left(|z|^2 e^{-\psi-\chi}\right)^2 + \left(1 - \frac{\kappa}{\mu}\right)\left(2e^{-\psi+\chi} - \frac{\xi}{M}\right)|z|^2 e^{-\psi-\chi} \\ &+ \left(1 + \frac{\kappa}{\mu}\right)^2\left(e^{-\psi+\chi}\right)^2 - \frac{\xi}{M}\left(1 + \frac{\kappa}{\mu}\right)e^{-\psi+\chi}\bigg] , \quad (11.77)\end{aligned}$$

which depends on $z, \bar{z}$ when the coupling constants are different, $\kappa \neq \mu$. However, when the coupling constants are equal, Eq. (11.77) reads

$$\bar{\partial}\partial\left(\psi - \chi\right) = -\frac{4\pi^2}{\kappa^2}\left(e^{-\psi+\chi} - \frac{\xi}{2M}\right)e^{-\psi+\chi} , \quad (11.78)$$

which allows the field combination $\psi - \chi$ to stay constant with the value

$$\psi - \chi = \log\left(\frac{2M}{\xi}\right) . \quad (11.79)$$

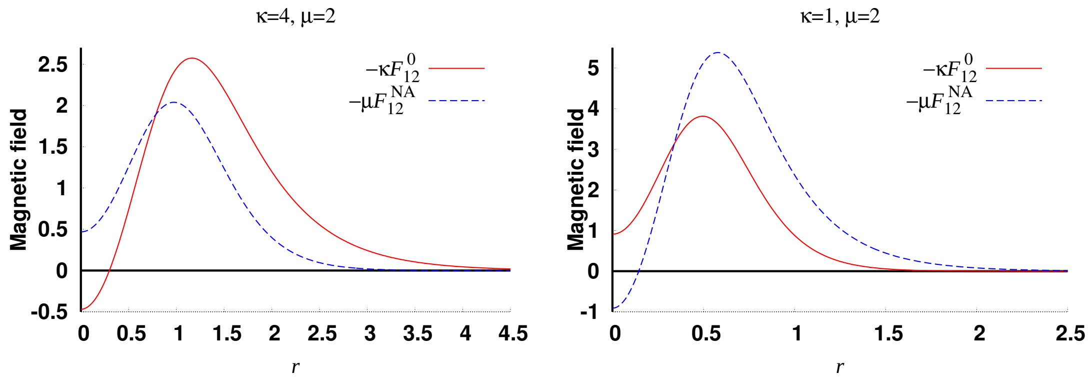
Plugging this (constant) solution into Eq. (11.76) we obtain readily $F^0_{12} = F^{\mathrm{NA}}_{12} = 0$ in the limit $r \to 0$.

In Fig. 11.3 is shown the Abelian (a) and non-Abelian (b) electric fields with different values of the couplings.

In Fig. 11.5 we show a sketch of the magnetic fields of Abelian and non-Abelian kinds, respectively, in the case of $\kappa > \mu$ (a) and in the case of $\kappa < \mu$ (b). The integral over the plane of the Abelian magnetic field density is proportional to the topological charge of the vortex, the winding number which in turn renders the soliton topologically stable. The vortex solution with negative winding number $k < 0$ can be interpreted as an anti-vortex. Hence, one could wonder which interpretation to give the small substructure found in this vortex solution – a small anti-vortex trapped in the non-Abelian vortex, as a bound state, not rendering the solution unstable.

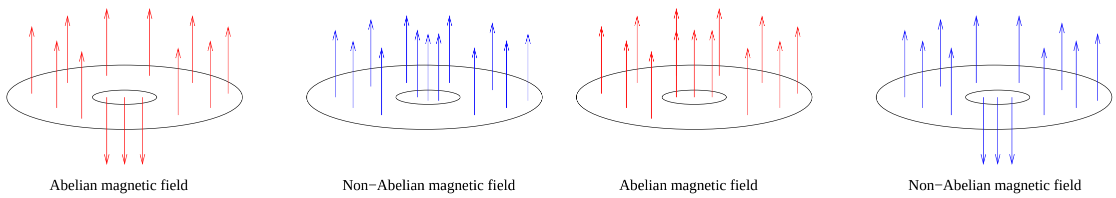
Opposite signs of coupling constants

We will now consider taking one of the couplings to be negative, say $\kappa < 0$ and $\mu > 0$. Choosing both signs negative yields the same solution as already mentioned, however with flipped electric fields. In the case of $\kappa > 0$ and $\mu < 0$, the solutions are equivalent to the ones we will consider now, just with the signs flipped of the electric fields. The Chern-Simons characteristics have been lost in this case, the vortex instead has the magnetic field concentrated at the origin – just as in the case of the ANO vortex or the single $U(N)$ non-Abelian generalization. In Fig. 11.6 the profile functions and energy densities for different solution are shown. In Fig. 11.7 the corresponding magnetic fields are shown while in Fig. 11.8 the electric fields are shown.



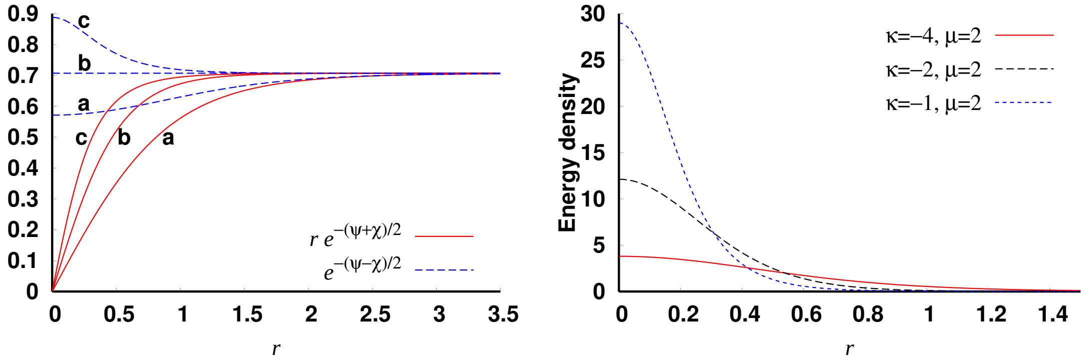
(a) (b)

Figure 11.4: Differently normalized Abelian and non-Abelian magnetic fields as κF^0_{12} and μF^{NA}_{12} for (a) $\kappa = 4, \mu = 2$ and (b) $\kappa = 1, \mu = 2$. This combination cancels exactly at the origin (to a numerical accuracy better than $\sim 10^{-5}$). The FI parameter $\xi = 2$.



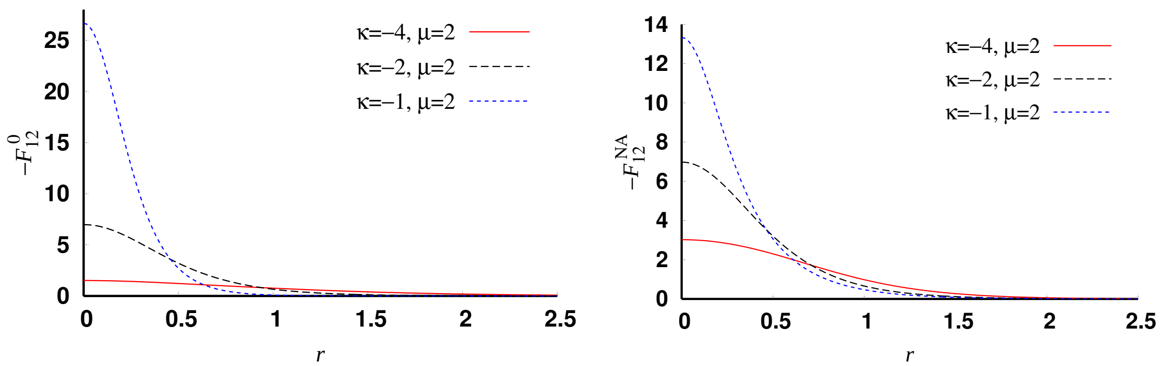
(a) $\kappa > \mu$ (b) $\kappa < \mu$

Figure 11.5: (a) Sketch of the magnetic fields where the Abelian (red/left) is negative at the origin and the non-Abelian (blue/right) is positive for $\kappa > \mu$. (b) Contrarily the Abelian (red/left) is positive at the origin while the non-Abelian (blue/right) is negative for $\kappa < \mu$.



(a) (b)

Figure 11.6: (a) Profile functions for three different values of the coupling constants; a: $\kappa = -4, \mu = 2$; b: $\kappa = -2, \mu = 2$; c: $\kappa = -1, \mu = 2$; the functions are plotted in traditional style with the winding field rising linearly and the non-winding field being constant at the origin. The FI parameter $\xi = 2$. Notice that the VEV for these functions is $2^{-\frac{1}{2}}$. (b) The energy density $\mathcal{E}$ for the vortex for the same three different values of the couplings with opposite signs. All the energy densities integrate to $\pi\xi$, within an accuracy better than $\sim 10^{-4}$. Notice that the extrema of the energy density is *at the origin*, just as in the case of the ANO vortices or the non-Abelian generalizations.



(a) (b)

Figure 11.7: (a) The Abelian magnetic field F^0_{12} (trace-part) and (b) the non-Abelian magnetic field $F^{\rm NA}_{12}$ (traceless part) for three different values of the couplings with opposite signs. Notice that the magnetic field density resembles that of the ANO vortex or the non-Abelian generalizations, viz. they have the extrema at the origin. The FI parameter $\xi = 2$.

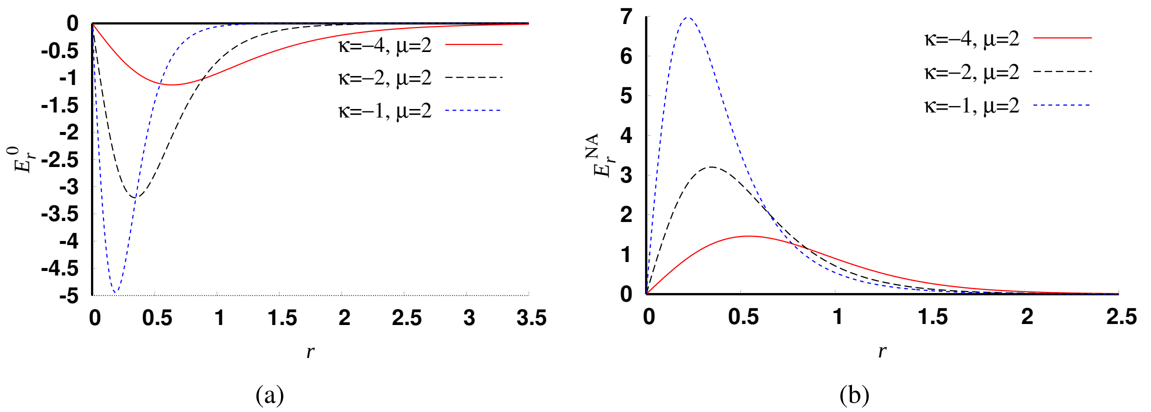


Figure 11.8: (a) The Abelian electric field in the radial direction E_r (trace-part) and (b) The non-Abelian electric field $E_r^{\rm NA}$ (traceless part) for three different values of the couplings with opposite signs. The FI parameter $\xi = 2$. Note that the electric fields are back-to-back.

11.6 Discussion

In this Chapter, we have thus brought the powerful moduli matrix formalism into the non-Abelian Chern-Simons model (which supports topological non-Abelian vortices), and have conjectured that the moduli spaces of the non-Abelian vortex solutions of these systems are indeed identical to those of the vortex solutions in the Yang-Mills-Higgs models with corresponding gauge groups. We have not proved that every moduli matrix has a unique and existing solution to the master equations found. Nevertheless we have argued the plausibility of such a claim by taking the weak coupling limit, which immediately yields the lumps of the Yang-Mills-Higgs models, as it is just the algebraic solutions to the D-term conditions.

We have then studied some numerical solutions of non-Abelian vortices, by choosing an Ansatz to the master equations, working mainly with the $G' = SO(4)$ and $G' = USp(4)$ gauge groups. We have studied the case of different couplings with both couplings positive yielding vortex solutions with a small negative Abelian (non-Abelian) magnetic field density at the origin and a corresponding positive non-Abelian (Abelian) magnetic field density, which have a combination that is always zero (at the origin). Keeping the couplings equal provides the typical Chern-Simons characteristic that the magnetic field vanishes at the origin yielding a ring structure. These new types of solutions could perhaps be interpreted as an anti-vortex sitting inside the non-Abelian vortex as a stable bound state, with the stability provided by topological arguments.

An interesting question is to which extent this substructure found in the non-Abelian vortex solutions alters the dynamics of the vortices.

Furthermore, by changing the relative sign of the coupling constants, a vortex solution with the magnetic field density concentrated at the origin has been found.

An obvious future study related to these vortices and also to the ones of Ref. [3] could be to make an explicit construction with exceptional groups and investigating the corresponding moduli spaces. Especially interesting would be the center-less groups.

Another interesting path to follow is to consider the construction of the non-Abelian vortices in Chern-Simons models with more supersymmetries, e.g. considering the model of Aharony-Bergman-Jafferis-Maldacena [262]. An Abelian non-relativistic Jackiw-Pi vortex has already been found in this model [263]. Another attempt to construct vortices in the latter model has recently been made, resulting in the non-Abelian vortex equations of the Yang-Mills-Higgs models [264].

Part VI

Final remarks

CHAPTER 12

Discussion and outlook

Let us summarize the most important results of this Thesis and conclude with an outlook. After we have given an introduction to the rich subject of solitons, and in particular vortices with references to the literature, we have introduced the notation and the many models which we work on and many of the equations have been derived step-by-step in Chap. 1.

The first direction of the Thesis was in the area of Abelian Chern-Simons vortices with focus on the large magnetic flux limit, where the well-known domain wall has been confirmed to describe the vortex and further used to find the type III vortex; first attractive and when accumulating flux to a certain point it becomes a repulsive cluster. Then an interesting observation for small Chern-Simons coupling constant, is that there should exist a smooth phase transition for very large magnetic flux.

The major part of this Thesis describes vortices and their associated strong coupling cousins, namely the corresponding lump solutions in theories with $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ gauge groups. The first main result is found within an Ansatz to solve the strong condition, i.e. for local vortices, where we formally find the quantization condition for the vortices which is the same as that of Goddard-Nuyts-Olive-Weinberg (GNOW) for monopoles. It is intriguing that it is exactly the dual group that comes out of this calculation. This is a further check for the ideas that the monopoles confined by vortices describe a (dual) superconductor as the confinement mechanism in the gauge theory in question. Studying in depth the moduli spaces of the local vortices we find the connectedness properties of the moduli spaces and represent the results graphically on the weight lattices of the dual groups. An important result is the topological disconnection given by the non-trivial $\pi_1(G')$, which we in these gauge theories have denoted the $\mathbb{Z}_2$ charge. This holds even when turning on semi-local moduli parameters. Another important result has come out from the studies of the semi-local vortices, namely that even for the minimal number of matter fields (multiplets) in order to obtain a color-flavor-locked type of vacuum, the vortex is semi-local if it is not restricted further by the so-called strong condition. This fact gives rise to the interest in studying the lumps in these theories, as they are rather general in the limit of strong coupling or low energies. The in-depth studies of the moduli spaces have been cross-checked with an index-theorem calculation agreeing with the number of moduli found within the moduli matrix formalism – a further non-trivial check to the assumption of existence and uniqueness properties of the master equations. In this arena we constructed as low-energy effective theories for the gauge theories in question, the Kähler quotients with the correct symmetry properties with explicit formal solutions, in the cases of $U(1) \times SO(2M)$, $U(1) \times USp(2M)$, $SO(2M)$ and $USp(2M)$. Furthermore, in the cases of $SO(2M), USp(2M)$ we have constructed also the hyper-Kähler quotients due to a trick using the algebra of the theories, which correspond to the low-energy effective theory for the $\mathcal{N} = 2$ theories. We discuss the vacuum properties of the theories and make an expansion formula for the result of the Kähler quotient calculation which we further use to obtain the scalar curvature and metric on the manifolds. Finally, we discuss the connection between the moduli spaces of vortices and the moduli spaces of lumps which turn out to be identical except for sick points not satisfying the so-called lump condition. We construct explicitly fractional lump solutions, the effective actions and identify the normalizable zero-modes of the single lump configurations.

The next part of the Thesis was concerned with the fractional vortices and makes a more detailed discussion of the mechanisms giving rise to the earlier found fractional lumps, especially using the simplest possible (BPS) Abelian gauge theories. An interesting result in this connection is related to the substructure of the solution and their dependence of the geometry of the target manifold.

The final part of this Thesis was dedicated to the non-Abelian Chern-Simons theories with generic gauge groups. Formally, the very powerful moduli matrix formalism has been ported to this kind of theories yielding interesting results about their moduli spaces and it was conjectured that they indeed are identical with the ones of the theories with only the Yang-Mills kinetic term, which have been studied in much more detail. Explicit solutions were obtained numerically providing new results for these theories with orthogonal and symplectic groups and furthermore a splitting of the magnetic flux was discovered. In the usual Abelian or non-Abelian theories with equal gauge coupling constants for the Abelian and the non-Abelian part, the magnetic flux is observed to vanish at the origin of the vortex solution. In the case of different gauge coupling, however, the magnetic fields (Abelian and non-Abelian) split yielding one magnetic field positive and another magnetic field negative, both non-vanishing. However, a linear combination of both of them is found always to vanish at the origin.

12.1 Outlook

We will here give a brief account of the further developments and ideas to continue this fascinating line of research described in this Thesis.

The first immediate idea to consider is the impact of the non-BPS corrections on the vortices and lumps described in this Thesis. Along this path comes the question of stability. Analytically, this can be done with our techniques in the near-BPS regime. For strong non-BPS-ness, we need to turn to numerical methods, which is a far more tedious step.

The low-energy effective vortex theory, which in the case of the $U(N)$ theories is the $\mathbb{C}P^{N-1}$ σ model, is not yet known for generic gauge groups. It would be very interesting to develop these effective theories, understand the vacuum structure and spectra of these theories and possibly confirm the coincidence with the spectra of the four-dimensional theories.

There are also several unanswered questions which are interesting and of importance. For instance, in the case of the Kähler quotients, we know that in the quantum regime, corrections have to kick in, and their importance surely cannot be underestimated. For the hyper-Kähler quotients we are in much better shape due to the non-renormalization theorems in supersymmetry, but we have not been able to resolve the explicit quotient in the case with a common Abelian factor. A further very interesting question is about the deformation of the Kähler potential in order to admit a Ricci-flat (non-compact Calabi-Yau) metric.

Higher representations of the gauge groups would be a very interesting development, and especially the adjoint representation of the matter fields, as it relates our theories with orthogonal groups and symplectic groups to multi-instanton moduli spaces in those theories.

Another interesting development is the extension of our program to the exceptional groups, which for instance in relation with the center-less groups could be very interesting in conjunction with lattice studies.

Certain mass-deformations of our theories could be very interesting for instance in conjunction with domain wall solutions. This could be interesting also if considering 1/4 BPS soliton junctions which furthermore are interesting in relation with cosmic string networks.

An explicit calculation of the second homotopy group for the orthogonal and symplectic groups is interesting for deepening the understanding of the fractional vortices appearing in these theories and for understanding the stratified geometry of the target space of the lump solutions.

For the lump solution, also here the non-BPS corrections are very interesting and for instance in a possible relation to the already found lumps supported by $\pi_2(\mathcal{M}) \simeq \mathbb{Z}_2$. Also the possibility of Q-lumps in the newly obtained Kähler quotients is quite interesting.

Let us mention a few developments already under study. One direction is related to the theory with both Chern-Simons and Yang-Mills kinetic term giving in-depth details about the underlying structure of the vortices. Another problem is about the complex of non-Abelian monopoles and vortices in the Higgs phase and the investigation of their mutual transformation properties.

Let us also mention the possibilities to port our techniques to the area of condensed matter physics, where new types of superconductors are being developed in the laboratory. This has amazing aspects for instance in relation with the realization of a non-Abelian superconductor, experimentally. Here an interesting and important subject to study are the so-called quantized vortices which have been found recently, which exhibit not only a non-trivial phase factor like normal vortices but combines it with a rotation of spin or orbital orientation. This type of vortices enjoys non-Abelian and non-commutative properties which are severely important for instance in collisions. Here so-called rung-vortices can arise in many cases, which are deeply rooted in the non-Abelian nature. These systems with their vortices are interesting because of the potential applications. Proposals for experiments are also possible here, for example in Bose-Einstein-Condensates (BECs), biaxial nematic liquid crystals and superconductors with high internal degrees of freedom, to just mention a few.

A further line of research that could have promising merits, is the understanding of non-BCS (Bardeen-Cooper-Schrieffer) superconductors, which actually exists in nature, mostly being made of composite materials and cuprates. These exotic kinds of superconductors are recently being explored using holographic superconductor techniques. A deeper understanding here would certainly be important.

Finally, interesting possibilities lie in the aspect of M-theory via the investigation of the M2-branes using Chern-Simons theory with a sufficient amount of supersymmetry. This is also interesting due to the existing gravity dual of these theories and not least because of the possible impact on the unraveling of the theory underlying string theory – viz. M-theory.

Part VII

Appendices

APPENDIX $\mathfrak{A}$

Generators of $SO(2N)$, $USp(2N)$ and $SO(2N+1)$

In a standard basis $J = \mathbf{1}_N$, the $SO(N)$ generators are given by

$$A \,, \tag{A.1}$$

where A are anti-symmetric imaginary matrices. Changing the base for $SO(2M)$ to the base with J given by Eq. (1.32) by a unitary transformation

$$O^{\mathrm{T}}O = 1, \quad O \to SOS^{-1} \,, \quad S = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1}_M & i\mathbf{1}_M \\ i\mathbf{1}_M & \mathbf{1}_M \end{pmatrix} \,, \tag{A.2}$$

yields the relation

$$O^{\mathrm{T}}JO = J \,, \quad iJ = S^{\mathrm{T}}S \,, \tag{A.3}$$

where J is exactly the invariant tensor (1.32). This is valid for elements of $SO(2M)$, but the same transformation can be applied to the generators yielding the constraint

$$t^{\mathrm{T}}J + Jt = 0 \,, \tag{A.4}$$

which is satisfied by

$$\begin{pmatrix} A & B_{\mathrm{A}} \\ B_{\mathrm{A}}^{\dagger} & -A^{\mathrm{T}} \end{pmatrix} \,, \tag{A.5}$$

where $B_{\mathrm{A}}^{\mathrm{T}} = -B_{\mathrm{A}}$ is anti-symmetric and $A^{\dagger} = A$ is Hermitian and both are complex matrices. Changing the tensor to the one for $USp(2M)$ in Eq. (1.32), the generators are on the form

$$\begin{pmatrix} A & B_{\mathrm{S}} \\ B_{\mathrm{S}}^{\dagger} & -A^{\mathrm{T}} \end{pmatrix} \,, \tag{A.6}$$

where $B_{\mathrm{S}}^{\mathrm{T}} = B_{\mathrm{S}}$ is symmetric and $A^{\dagger} = A$ is Hermitian and both are complex matrices. In the case of $SO(2M+1)$, the generators with the invariant tensor of Eq. (1.32) are of the form

$$\begin{pmatrix} A & B_{\mathrm{A}} & f^* \\ B_{\mathrm{A}}^{\dagger} & -A^{\mathrm{T}} & f \\ -f^{\mathrm{T}} & -f^{\dagger} & 0 \end{pmatrix} \,, \tag{A.7}$$

where $B_{\mathrm{A}}^{\mathrm{T}} = -B_{\mathrm{A}}$ is anti-symmetric and $A^{\dagger} = A$ is Hermitian and both are complex matrices, while f is a complex vector.

It will be useful to also state the complexified generators in this basis (1.32). In the case of $SO(2M)^{\mathbb{C}}$, $USp(2M)^{\mathbb{C}}$ we have

$$\begin{pmatrix} A & B_{\mathrm{A,S}} \\ C_{\mathrm{A,S}} & -A^{\mathrm{T}} \end{pmatrix} , \tag{A.8}$$

where subscript A, S is for anti-symmetric and symmetric, for $SO(2M)^{\mathbb{C}}$ and $USp(2M)^{\mathbb{C}}$, respectively. All the matrices $A, B_{\mathrm{A,S}}, C_{\mathrm{A,S}}$ are complex. In the case of $SO(2M+1)^{\mathbb{C}}$ we have

$$\begin{pmatrix} A & B_{\mathrm{A}} & f \\ C_{\mathrm{A}} & -A^{\mathrm{T}} & g \\ -g^{\mathrm{T}} & -f^{\mathrm{T}} & 0 \end{pmatrix} , \tag{A.9}$$

where $B_{\mathrm{A}}^{\mathrm{T}} = -B_{\mathrm{A}}$ and $C_{\mathrm{A}}^{\mathrm{T}} = -C_{\mathrm{A}}$ are anti-symmetric, $A, B_{\mathrm{A}}, C_{\mathrm{A}}$ are complex matrices and f, g are complex vectors.

APPENDIX $\mathfrak{B}$

Spatially separated vortices

When the two vortices are separated, i.e. $\delta \neq 0$, the second equation of Eq. (5.46) (together with $\mathrm{Tr}\,\Gamma = 0$) is solved by

$$\Gamma = o'\,\Gamma_0\,o'^{-1}, \qquad \Gamma_0 \equiv \sqrt{\delta}\begin{pmatrix} \mathbf{1}_{M-r} & \\ & -\mathbf{1}_{M-r} \end{pmatrix} . \tag{B.1}$$

There remains an arbitrariness under reshuffling of the form,

$$o' \to o's\ , \quad \Gamma_0 \to s^{-1}\Gamma_0 s\ , \quad s \equiv \begin{pmatrix} u'_1 & \\ & u'_2 \end{pmatrix} , \tag{B.2}$$

where $u'_i \in GL(M-r,\mathbb{C})$. Then the first condition in Eq. (5.46) leads to

$$o'^{\mathrm{T}} J_{2(M-r)}\,o' = \begin{pmatrix} 0 & X \\ \epsilon\,X^{\mathrm{T}} & 0 \end{pmatrix} \sim J_{2(M-r)}\ , \tag{B.3}$$

where we have used the above-mentioned freedom to arrive at the last form for $J_{2(M-r)}$. The above relation means that o' is an element of $O(2(M-r))^{\mathbb{C}}$ ($USp(2(M-r))^{\mathbb{C}}$). There exists still an unphysical transformation

$$u_1'^{\,\mathrm{T}} = u_2'^{\,-1} \equiv u \in GL(M-r,\mathbb{C})\ . \tag{B.4}$$

Thus the solution of the strong condition (5.46) with $\delta \neq 0$ is given by

$$\Gamma \in \left\{ \begin{array}{ll} \left\{\mathbb{C}^* \times \left[\frac{O(2(M-r),\mathbb{C})}{U(M-r)}\right]^{\mathbb{C}}\right\}/\mathbb{Z}_2 & \text{for} \quad G' = SO(2M)\ , \\ \left\{\mathbb{C}^* \times \left[\frac{USp(2(M-r),\mathbb{C})}{U(M-r)}\right]^{\mathbb{C}}\right\}/\mathbb{Z}_2 & \text{for} \quad G' = USp(2M)\ , \end{array} \right. \tag{B.5}$$

with the first $\mathbb{C}^*$ factor being the relative distance $\sqrt{\delta}$. The $\mathbb{Z}_2$ factors in the denominators come about due to the fact that a combination of a π-rotation in the x^1–x^2 space $\sqrt{\delta} \to -\sqrt{\delta}$ and a permutation $o' \to o'p$, satisfying $p\Gamma_0 p^{-1} = -\Gamma_0$ is an identity operation.

APPENDIX $\mathfrak{C}$

Fixing NG modes for Sec. 5.4.1

Let us go into a detailed investigation, in order to verify the results in Sec. 5.4.1. In the first place note that $a_{0;A,S}$ and $C_{1,2}$ are obviously NG modes when two vortices are coincident, namely $\delta = 0$. One can confirm this fact, for example, by considering an infinitesimal color-flavor G'_{C+F} transformation accompanied by an appropriate V-transformation. Therefore, any moduli matrix of the form (5.41) can be always brought into the following

$$H_0^{(\overbrace{1,\cdots,1}^{r},\overbrace{0,\cdots,0}^{M-r})} = \begin{pmatrix} (z-z_0)^2\mathbf{1}_r & 0 & 0 & 0 \\ 0 & (z-z_0)\mathbf{1}_{M-r}+\Gamma_{11} & 0 & \Gamma_{12} \\ a_{1;A,S}\, z & 0 & \mathbf{1}_r & 0 \\ 0 & \Gamma_{21} & 0 & (z-z_0)\mathbf{1}_{M-r}+\Gamma_{22} \end{pmatrix} . \tag{C.1}$$

For $\delta = 0$, the rank $2\gamma = \mathrm{rank}(\Gamma)$ is less than $2\gamma < 2(M-r)$. The first condition in Eq. (5.46) states that $\Gamma J_{2(M-r)}$ is anti-symmetric (symmetric), so that Γ can be written as

$$\Gamma = \epsilon\, q \tilde{J}_{2\gamma}\, q^{\mathrm{T}}\, J_{2(M-r)} \;, \tag{C.2}$$

where q is a $2(M-r)\times 2\gamma$ matrix whose rank is 2γ, $(M-r\geq\gamma)$, and $\tilde{J}_{2\gamma}$ is the invariant tensor of $\tilde{G}'_{2\gamma} = USp(2\gamma)$ for $G' = SO(2M)$ and $\tilde{G}'_{2\gamma} = SO(2\gamma)$ for $G' = USp(2M)$. Then the second condition is translated into the following constraint on q:

$$A = 0\;, \quad A \equiv q^{\mathrm{T}} J_{2(M-r)} q\;. \tag{C.3}$$

Note that the rank of $A = q^{\mathrm{T}} J_{2(M-r)} q$ is bounded as

$$4\gamma - 2(M-r) \leq \mathrm{rank}(A) \leq \mathrm{rank}(q) = 2\gamma\;. \tag{C.4}$$

Therefore, $2\gamma \leq M-r$ in the present case of $\mathrm{rank}(A) = 0$. This last condition can be solved by

$$q = O\begin{pmatrix} g \\ \mathbf{0}_{2(M-r-\gamma)\times 2\gamma} \end{pmatrix}\;, \quad g \in GL(2\gamma,\mathbb{C})\;, \quad O \in G'_{2(M-r)}\;. \tag{C.5}$$

Thus we find

$$\Gamma = O \left(\begin{array}{c|c} \begin{matrix} g\tilde{J}_{2\gamma}g^{\mathrm{T}} & \\ & \mathbf{0}_{M-r-2\gamma} \end{matrix} & \\ \hline & \begin{matrix} \mathbf{0}_{2\gamma} & \\ & \mathbf{0}_{M-r-2\gamma} \end{matrix} \end{array}\right) O^{\mathrm{T}} J_{2(M-r)}\;. \tag{C.6}$$

In the case of $G' = SO(2M)$, we can bring the anti-symmetric matrix $g\tilde{J}_{2\gamma}g^{\mathrm{T}}$ onto a block-diagonal form as

$$g\tilde{J}_{2\gamma}g^{\mathrm{T}} = u\,\Lambda\,u^{\mathrm{T}}\ , \quad \Lambda \equiv i\sigma_2 \otimes \mathrm{diag}(\lambda_1\mathbf{1}_{p_1}, \lambda_2\mathbf{1}_{p_2}, \cdots, \lambda_q\mathbf{1}_{p_q})\ , \quad (\lambda_i > \lambda_{i+1} > 0)\ , \tag{C.7}$$

where $u \in U(2\gamma)$ and $2\sum_{i=1}^{q} p_i = 2\gamma$. Thus we have found

$$\Gamma \;=\; O' \left(\begin{array}{cc|cc} & & \Lambda & \\ & & & \mathbf{0}_{M-r-2\gamma} \\ \hline \mathbf{0}_{2\gamma} & & & \\ & \mathbf{0}_{M-r-2\gamma} & & \end{array}\right) O'^{-1}\ , \tag{C.8}$$

$$O' \;\equiv\; O \begin{pmatrix} u & & & \\ & \mathbf{1}_{M-r-2\gamma} & & \\ & & \left(u^{\mathrm{T}}\right)^{-1} & \\ & & & \mathbf{1}_{M-r-2\gamma} \end{pmatrix} \in SO(2(M-r))\ . \tag{C.9}$$

Similarly, the anti-symmetric tensor $a_{1;,A}$ can be brought onto a diagonal form. Let

$$\mathrm{rank}(a_{1,A}) = 2\alpha \leq r\ ,$$

then we obtain

$$a_{1;A} = \begin{pmatrix} \mathbf{0}_{r-\alpha} & \\ & u'\,\Lambda'\,u'^{\mathrm{T}} \end{pmatrix}\ , \quad \Lambda' \equiv i\sigma_2 \otimes \mathrm{diag}(\lambda'_1\mathbf{1}_{p'_1}, \lambda'_2\mathbf{1}_{p'_2}, \cdots, \lambda'_{q'}\mathbf{1}_{p'_{q'}})\ , \tag{C.10}$$

where $u' \in U(2\alpha)$, $2\sum_{i=1}^{q'} p'_i = 2\alpha$ and $\lambda'_i > \lambda'_{i+1} > 0$. Finally, we arrive at the following expression

$$H_0 = \left(\begin{array}{cc|cc||cc|cc} z^2\mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2\mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & z\mathbf{1}_{2\gamma} & 0 & 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & \mathbf{0}_{M-r-2\gamma} \\ \hline\hline \mathbf{0}_{r-2\alpha} & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 \\ 0 & \Lambda' z & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{2\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} \end{array}\right)\ , \tag{C.11}$$

where we have turned off the center of mass $z_0 = 0$. One can return to the previous moduli matrix by using the color-flavor symmetry $H_0 \to U^{-1}H_0U$ with

$$U \equiv \left(\begin{array}{cc|c||cc|c} \mathbf{1}_{r-2\alpha} & & & & & \\ & u'^{\mathrm{T}} & & & & \\ \hline & & O'^{-1} & & & \\ \hline\hline & & & \mathbf{1}_{r-2\alpha} & & \\ & & & & u'^{-1} & \\ \hline & & & & & O'^{-1} \end{array}\right) \in SO(2M)\ . \tag{C.12}$$

By making use of the V-transformation, one can bring this onto the following form

$$VH_0 = \left(\begin{array}{cc|cc||cc|cc} z^2\mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2\mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & z^2\mathbf{1}_{2\gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 \\ 0 & \Lambda' z & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 \\ \hline 0 & 0 & \Lambda^{-1} z & 0 & 0 & 0 & \mathbf{1}_{2\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} \end{array}\right),$$

$$V = \left(\begin{array}{cc|cc||cc|cc} \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & z\mathbf{1}_{2\gamma} & 0 & 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & \mathbf{1}_{M-r-2\gamma} & 0 & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & \mathbf{1}_{r-2\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{2\alpha} & 0 & 0 \\ \hline 0 & 0 & \Lambda^{-1} & 0 & 0 & 0 & \mathbf{0}_{2\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-2\gamma} \end{array}\right). \tag{C.13}$$

where one can check that $V \in SO(2M, \mathbb{C})$ because $\Lambda^{\mathrm{T}} = -\Lambda$. We can rearrange the eigenvalues $\tilde{\lambda}_a = \{\lambda_i^{-1}, \lambda'_j\}$ in such a way that

$$\mathrm{diag}\left(\Lambda',\ \Lambda^{-1}\right) = i\sigma_2 \otimes \mathrm{diag}\left(\tilde{\lambda}_1 \mathbf{1}_{\tilde{p}_1}, \cdots, \tilde{\lambda}_s \mathbf{1}_{\tilde{p}_s}\right), \quad \tilde{\lambda}_a > \tilde{\lambda}_{a+1} > 0, \tag{C.14}$$

hence the $G'_{\mathrm{C+F}} = SO(2M)$ orbit can easily be seen in Eq. (5.70).

The arguments for $G' = USp(2M)$ are analogous to those of $G' = SO(2M)$. A small difference is that $J_{2(M-r)}\Gamma$ and $a_{1;S}$ are now symmetric. In the end, we obtain the moduli matrix on the following form

$$H_0 = \left(\begin{array}{ccc|ccc} z^2\mathbf{1}_{r-\beta} & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2\mathbf{1}_{\beta+\zeta} & 0 & 0 & 0 & 0 \\ 0 & 0 & z\mathbf{1}_{M-r-\zeta} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{1}_{r-\beta} & 0 & 0 \\ 0 & \tilde{\Lambda} z & 0 & 0 & \mathbf{1}_{\beta+\zeta} & 0 \\ 0 & 0 & 0 & 0 & 0 & z\mathbf{1}_{M-r-\zeta} \end{array}\right), \tag{C.15}$$

$$\tilde{\Lambda} = \mathrm{diag}(\tilde{\lambda}_1 \mathbf{1}_{\tilde{p}_1}, \cdots, \tilde{\lambda}_s \mathbf{1}_{\tilde{p}_s}), \tag{C.16}$$

with $\beta = \mathrm{rank}(\Gamma)$ and $\zeta = \mathrm{rank}(a_{1;S})$.

APPENDIX $\mathfrak{D}$

Some transition functions for semi-local vortices

Here we make a collection of some of the transition functions discussed in Chaps. 5 and 6.

D.1 $k = 1$, $G' = SO(4)$

The transition functions between two $\mathbb{Z}_2$-parity $+1$ patches for the minimal semi-local vortices in $G' = SO(4)$ theory of Sec. 6.4.1:

$$\begin{cases} a = -f'i' + \frac{a'+d'}{2} \ , \\ b = -g'i' \ , \\ c = e'i' \ , \\ d = f'i' + \frac{a'+d'}{2} \ , \\ e = -c'i' \ , \\ f = \frac{(a'-d')i'}{2} \ , \\ g = b'i' \ , \\ i = -\frac{1}{i'} \ . \end{cases} \tag{D.1}$$

D.2 $k = 2$, $G' = SO(4)$, $Q_{\mathbb{Z}_2} = +1$

The transition functions between $H_0^{(0,0)}$ and $H_0^{(1,1)}$ for the $k = 2$ semi-local vortices in $G' = SO(4)$ theory of Sec. 6.5.1:

$$\begin{cases} a_0 &= \frac{1}{2}a_1' - \frac{1}{2}d_1' + \frac{i_0'}{i_1'} \ , \\ b_0 &= b_1' \ , \\ c_0 &= e_1' \ , \\ d_0 &= f_1' - \frac{1}{i_1'} \ , \\ e_0 &= c_1' \ , \\ f_0 &= -\frac{1}{2}a_1' + \frac{1}{2}d_1' + \frac{i_0'}{i_1'} \ , \\ g_0 &= f_1' + \frac{1}{i_1'} \ , \\ h_0 &= g_1' \ , \\ i_0 &= -c_1' i_0' - c_0' i_1' + \frac{1}{2}a_1' c_1' i_1' + \frac{1}{2}c_1' d_1' i_1' \ , \\ j_0 &= a_1' i_0' - \frac{i_0'^2}{i_1'} - \frac{1}{4}a_1'^2 i_1' - d_0' i_1' + \frac{1}{4}d_1'^2 i_1' \ , \\ k_0 &= \frac{1}{2}a_1' + \frac{1}{2}d_1' - f_1' i_0' - f_0' i_1' - \frac{i_0'}{i_1'} + \frac{1}{2}a_1' f_1' i_1' + \frac{1}{2}d_1' f_1' i_1' \ , \\ l_0 &= -g_1' i_0' - g_0' i_1' + \frac{1}{2}a_1' g_1' i_1' + \frac{1}{2}d_1' g_1' i_1' \ , \\ m_0 &= -d_1' i_0' + a_0' i_1' + \frac{i_0'^2}{i_1'} - \frac{1}{4}a_1'^2 i_1' + \frac{1}{4}d_1'^2 i_1' \ , \\ n_0 &= b_1' i_0' + b_0' i_1' - \frac{1}{2}a_1' b_1' i_1' - \frac{1}{2}d_1' b_1' i_1' \ , \\ o_0 &= e_1' i_0' + e_0' i_1' - \frac{1}{2}a_1' e_1' i_1' - \frac{1}{2}d_1' e_1' i_1' \ , \\ p_0 &= \frac{1}{2}a_1' + \frac{1}{2}d_1' + f_1' i_0' + f_0' i_1' - \frac{i_0'}{i_1'} - \frac{1}{2}a_1' f_1' i_1' - \frac{1}{2}d_1' f_1' i_1' \ . \end{cases} \tag{D.2}$$

These transition functions are, of course, invertible.

D.3 $k=2$, $G'=SO(4)$, $Q_{\mathbb{Z}_2}=-1$

The transition functions between the patches with $\mathbb{Z}_2$-parity -1, viz. $H_0^{(1,0)}$ and $H_0^{(-1,0)}$, for the $k=2$ semi-local vortices in $G'=SO(4)$ theory discussed in Sec. 6.5.1, are

$$\begin{cases} a_1 = -c_1' e_1' i_1' - \frac{e_0'}{e_1'} - \frac{i_0'}{i_1'} \,, \\ a_0 = -c_0' e_1' i_1' + \frac{e_0' i_0'}{e_1' i_1'} \,, \\ b_1 = -b_1' e_1' i_1' - f_0' i_1' - e_1' j_0' + \frac{e_0' i_1'}{e_1'} \,, \\ b_0 = -b_0' e_1' i_1' - f_0' i_0' - e_0' j_0' + \frac{e_0' i_0'}{e_1'} \,, \\ c_1 = -a_1' e_1' i_1' - e_0' i_1' - e_1' i_0' \,, \\ c_0 = -a_0' e_1' i_1' + e_0' i_0' \,, \\ d_1 = -d_1' e_1' i_1' - g_0' i_1' - e_1' k_0' - \frac{e_1' i_0'}{i_1'} \,, \\ d_0 = -d_0' e_1' i_1' + g_0' i_0' + e_0' k_0' - \frac{e_0' i_0'}{i_1'} \\ e_1 = -\frac{1}{i_1'} \,, \\ e_0 = -\frac{i_0'}{i_1'^2} \,, \\ f_0 = -\frac{i_0'}{i_1'} - \frac{e_1' j_0'}{i_1'} \,, \\ g_0 = -\frac{e_1' k_0'}{i_1'} - \frac{e_1' i_0'}{i_1'^2} \,, \\ i_1 = -\frac{1}{e_1'} \,, \\ i_0 = -\frac{e_0'}{e_1'^2} \,, \\ j_0 = -\frac{i_1' f_0'}{e_1'} - \frac{i_1' e_0'}{e_1'^2} \,, \\ k_0 = -\frac{g_0' i_1'}{e_1'} - \frac{e_0'}{e_1'} \,. \end{cases} \tag{D.3}$$

D.4 $k=1$, $G'=SO(3)$

The transition functions between the patches (-1) and (1) for the $k=1$ semi-local vortices in $G'=SO(3)$ theory discussed in Sec. 6.6.1, are

$$\begin{cases} d = -\frac{2}{d'} \,, \\ e = -\frac{2e'}{d'^2} \,, \\ z_3 = -\frac{2e'}{d'} - z_3' \,, \\ f = d'e' - \frac{1}{2} d'^2 z_1' \,, \\ a = \frac{1}{2}\left(e'^2 - d'^2 z_2'\right) \,, \\ b = -\frac{1}{2} b' d'^2 - e' - d' z_3' \,, \\ c = -\frac{1}{2} c' d'^2 - e'\left(\frac{e'}{d'} + z_3'\right) \,, \\ z_1 = \frac{2e'}{d'} - \frac{1}{2} d'^2 f' \,, \\ z_2 = \frac{e'^2}{d'^2} - \frac{1}{2} a' d'^2 \,. \end{cases} \tag{D.4}$$

D.5 $k=1, G'=SO(5), Q_{\mathbb{Z}_2}=+1$

The transition functions between the patches $(1,1)$ and $(0,0)$ for the $k=1$ semi-local vortices in $G'=SO(5)$ theory discussed in Sec. 6.6.2, are

$$\begin{cases} a_1' = \frac{a_1-a_4}{2} + \frac{f}{e} + \frac{i_1 i_2}{2e} \ , \\ a_2' = a_2 + \frac{i_2^2}{2e} \ , \\ a_3' = c_1 \ , \\ a_4' = -\frac{1}{e} + c_2 \ , \\ a_5' = g_1 - \frac{i_2}{e} \ , \\ b_1' = a_3 - \frac{i_1^2}{2e} \ , \\ b_2' = -\frac{a_1-a_4}{2} + \frac{f}{e} - \frac{i_1 i_2}{2e} \ , \\ b_3' = \frac{1}{e} + c_2 \ , \\ b_4' = c_3 \ , \\ b_5' = g_2 + \frac{i_1}{e} \ , \\ c_1' = -eb_3 + \frac{ea_3(a_1+a_4)}{2} - a_3 f - \frac{i_1(a_1 i_1 + a_3 i_2)}{2} - i_1 j_1 \ , \\ c_2' = -eb_4 - \frac{e\left(a_1^2-a_4^2\right)}{4} + a_1 f - \frac{f^2}{e} - \frac{i_1(a_2 i_1 + a_4 i_2)}{2} - i_1 j_2 \ , \\ c_3' = -ed_2 + \frac{c_2 e(a_1+a_4)}{2} + \frac{a_1+a_4}{2} - \frac{f}{e} - c_2 f - \frac{i_1(c_1 i_1 + c_2 i_2)}{2} - \frac{i_1 i_2}{2e} \ , \\ c_4' = -ed_3 + \frac{c_3 e(a_1+a_4)}{2} - c_3 f + \frac{i_1^2}{2e} - \frac{i_1(c_2 i_1 + c_3 i_2)}{2} \ , \\ c_5' = -eh_2 + \frac{g_2 e(a_1+a_4)}{2} - f g_2 + \frac{i_1(a_1+a_4)}{2} - \frac{f i_1}{e} - \frac{i_1(g_1 i_1 + g_2 i_2)}{2} - i_1 y \ , \\ d_1' = eb_1 - \frac{e\left(a_1^2-a_4^2\right)}{4} - a_4 f + \frac{f^2}{e} - \frac{i_2(a_1 i_1 + a_3 i_2)}{2} - i_2 j_1 \ , \\ d_2' = eb_2 - \frac{a_2 e(a_1+a_4)}{2} + a_2 f - \frac{i_2(a_2 i_1 + a_4 i_2)}{2} - i_2 j_2 \ , \\ d_3' = ed_1 - \frac{c_1 e(a_1+a_4)}{2} + c_1 f - \frac{i_2(c_1 i_1 + c_2 i_2)}{2} - \frac{i_2^2}{2e} \ , \\ d_4' = ed_2 - \frac{c_2 e(a_1+a_4)}{2} + \frac{a_1+a_4}{2} - \frac{f}{e} + c_2 f - \frac{i_2(c_2 i_1 + c_3 i_2)}{2} + \frac{i_1 i_2}{2e} \ , \\ d_5' = eh_1 - \frac{g_1 e(a_1+a_4)}{2} + f g_1 + \frac{i_2(a_1+a_4)}{2} - \frac{f i_2}{e} - \frac{i_2(g_1 i_1 + g_2 i_2)}{2} - i_2 y \ , \\ e_1' = j_1 + \frac{i_1(a_1-a_4)}{2} + \frac{f i_1}{e} + a_3 i_2 \ , \\ e_2' = j_2 - \frac{i_2(a_1-a_4)}{2} + \frac{f i_2}{e} + a_2 i_1 \ , \\ e_3' = c_1 i_1 + c_2 i_2 + \frac{i_2}{e} \ , \\ e_4' = c_2 i_1 + c_3 i_2 - \frac{i_1}{e} \ , \\ e_5' = y + g_1 i_1 + g_2 i_2 \ , \end{cases} \tag{D.5}$$

D.6 $k = 1$, $G' = SO(5)$, $Q_{\mathbb{Z}_2} = -1$

The transition functions between the patches $(-1, 0)$ and $(1, 0)$ for the $k = 1$ semi-local vortices in $G' = SO(5)$ theory discussed in Sec. 6.6.2, are

$$
\left\{
\begin{aligned}
a_0 &= \tfrac{2f'_0f'_1+h'^2_0}{\Xi} - \tfrac{1}{2}b'_0\Xi\,, \\
a_1 &= -\tfrac{2\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi} - \tfrac{1}{2}b'_1\Xi\,, \\
b_0 &= f'_0f'_1 + \tfrac{1}{2}h'^2_0 - \tfrac{1}{2}a'_0\Xi\,, \\
b_1 &= -e'_0f'_0 - e'_1f'_1 - h'_0j'_1 - \tfrac{1}{2}a'_1\Xi\,, \\
c_0 &= f'_1g'_0 + f'_0g'_2 + h'_0h'_1 + \tfrac{e'_0\left(2f'_0f'_1+h'^2_0\right)}{\Xi} - \tfrac{1}{2}c'_0\Xi\,, \\
c_1 &= f'_1 - e'_0g'_0 - e'_1g'_2 - h'_1j'_1 - \tfrac{2e'_0\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi} - \tfrac{1}{2}c'_1\Xi\,, \\
d_0 &= f'_1g'_1 + f'_0g'_3 + h'_0h'_2 + \tfrac{e'_1\left(2f'_0f'_1+h'^2_0\right)}{\Xi} - \tfrac{1}{2}d'_0\Xi\,, \\
d_1 &= f'_0 - e'_0g'_1 - e'_1g'_3 - h'_2j'_1 - \tfrac{2e'_1\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi} - \tfrac{1}{2}d'_1\Xi\,, \\
e_0 &= -\tfrac{2e'_0}{\Xi}\,, \\
e_1 &= -\tfrac{2e'_1}{\Xi}\,, \\
f_0 &= \tfrac{-4e'^2_1f'_1-4e'_1h'_0j'_1+2f'_0j'^2_1}{\Xi^2}\,, \\
f_1 &= \tfrac{-4e'^2_0f'_0-4e'_0h'_0j'_1+2f'_1j'^2_1}{\Xi^2}\,, \\
g_0 &= g'_0 - \tfrac{2e'_1\left(e'_0g'_0+e'_1g'_2+h'_1j'_1\right)}{\Xi} + \tfrac{2e'_0\left(-2e'^2_1f'_1-2e'_1h'_0j'_1+f'_0j'^2_1\right)}{\Xi^2}\,, \\
g_1 &= \tfrac{4e'^3_1\left(f'_1+e'_0g'_3\right)+g'_1j'^4_1-2e'^2_1j'_1\left(2h'_0+2e'_0h'_2+g'_3j'_1\right)+2e'_1j'^2_1\left(f'_0+e'_0g'_1-h'_2j'_1\right)}{\Xi^2}\,, \\
g_2 &= \tfrac{-4e'^3_0\left(f'_0+e'_1g'_0\right)+g'_2j'^4_1-2e'^2_0j'_1\left(2h'_0+2e'_1h'_1+g'_0j'_1\right)+2e'_0j'^2_1\left(f'_1+e'_1g'_2-h'_1j'_1\right)}{\Xi^2}\,, \\
g_3 &= g'_3 - \tfrac{4e'_0e'_1\left(e'_0f'_0+e'_1f'_1+h'_0j'_1\right)}{\Xi^2} - \tfrac{2\left(-e'_1f'_1+e'^2_0g'_1+e'_0e'_1g'_3+e'_0h'_2j'_1\right)}{\Xi}\,, \\
h_0 &= \tfrac{-4e'_0e'_1h'_0+4e'_0f'_0j'_1+4e'_1f'_1j'_1+2h'_0j'^2_1}{\Xi^2}\,, \\
h_1 &= \tfrac{4e'^2_0\left(-e'_1\left(h'_0+e'_1h'_1\right)+\left(f'_0+e'_1g'_0\right)j'_1\right)+j'^3_1\left(2e'_1g'_2+h'_1j'_1\right)+2e'_0j'_1\left(2e'_1\left(f'_1+e'_1g'_2\right)+j'_1\left(h'_0+g'_0j'_1\right)\right)}{\Xi^2}\,, \\
h_2 &= \tfrac{-4e'_0e'^2_1\left(h'_0+e'_0h'_2\right)+4e'_1\left(e'_1f'_1+e'_0\left(f'_0+e'_0g'_1+e'_1g'_3\right)\right)j'_1+2e'_1h'_0j'^2_1+2\left(e'_0g'_1+e'_1g'_3\right)j'^3_1+h'_2j'^4_1}{\Xi^2}\,, \\
i_0 &= \tfrac{2e'^2_0e'^2_1i'_0-2e'_0e'_1\left(f'_1j'_0-i'_0j'^2_1+f'_0j'_2+h'_0k'\right)+j'_1\left(-f'_1j'_0j'_1+\frac{1}{2}i'_0j'^3_1-f'_0\left(2f'_1+j'_1j'_2\right)-h'_0\left(h'_0+j'_1k'\right)\right)}{\Xi}\,, \\
i_1 &= \tfrac{2e'^2_0e'_1\left(e'_1i'_1+j'_0\right)+j'_1\left(2e'_1f'_1+h'_0j'_1+\frac{1}{2}i'_1j'^3_1+e'_1j'_1j'_2+j'^2_1k'\right)}{\Xi} \\
&\quad + \tfrac{e'_0\left(j'_1\left(2f'_0+j'_0j'_1\right)+2e'^2_1j'_2+2e'_1\left(-h'_0+j'_1\left(i'_1j'_1+k'\right)\right)\right)}{\Xi}\,, \\
j_0 &= \tfrac{-j'^3_1\left(2f'_0+j'_0j'_1\right)+4e'_0e'^3_1j'_2+2e'^2_1j'_1\left(2f'_1+j'_1j'_2+2e'_0k'\right)+2e'_1j'^2_1\left(2h'_0-e'_0j'_0+j'_1k'\right)}{\Xi^2}\,, \\
j_1 &= \tfrac{2j'_1}{\Xi}\,, \\
j_2 &= \tfrac{4e'^3_0e'_1j'_0-j'^3_1\left(2f'_1+j'_1j'_2\right)+2e'^2_0j'_1\left(2f'_0+j'_0j'_1+2e'_1k'\right)+2e'_0j'^2_1\left(2h'_0-e'_1j'_2+j'_1k'\right)}{\Xi^2}\,, \\
k &= \tfrac{-2e'_0j'_1\left(-2e'_1h'_0+j'_1\left(2f'_0+j'_0j'_1\right)+2e'^2_1j'_2\right)+4e'^2_0e'_1\left(-j'_0j'_1+e'_1k'\right)-j'^2_1\left(4e'_1f'_1+2h'_0j'_1+2e'_1j'_1j'_2+j'^2_1k'\right)}{\Xi^2}\,.
\end{aligned}
\right.
\tag{D.6}
$$

APPENDIX $\mathfrak{E}$

$SO(2M), USp(2M)$ groups and their invariant tensors

Let us define the following sets of n-by-n matrices for $\epsilon = \pm 1$

$$\mathrm{Inv}_\epsilon(n) \equiv \{J \mid J^{\mathrm{T}} = \epsilon J\ ,\ J^\dagger J = \mathbf{1}_n\}\ . \tag{E.1}$$

That is, elements of $\mathrm{Inv}_\epsilon(n)$ are (anti)symmetric and unitary.
Proposition: For an arbitrary $A \in \mathrm{Inv}_+(2)$, there exists a 2-by-2 unitary matrix u such that

$$A = u^{\mathrm{T}} u\ . \tag{E.2}$$

Proof: A general solution of A is given by

$$\begin{aligned} A &= e^{i\lambda}\begin{pmatrix} e^{i\rho}\cos\theta & i\sin\theta \\ i\sin\theta & e^{-i\rho}\cos\theta \end{pmatrix} \\ &= e^{\frac{i}{2}(\lambda\mathbf{1}_2+\rho\sigma_3)}\left(\cos\theta\mathbf{1}_2 + i\sigma_1\sin\theta\right)e^{\frac{i}{2}(\lambda\mathbf{1}_2+\rho\sigma_3)} = u^{\mathrm{T}}u\ , \end{aligned} \tag{E.3}$$

with $u = e^{\frac{i}{2}\theta\sigma_1}e^{\frac{i}{2}(\lambda\mathbf{1}_2+\rho\sigma_3)} \in U(2)$. ■
Theorem 1-s: An arbitrary $A \in \mathrm{Inv}_+(n)$ can be written as

$$A = u^{\mathrm{T}} u\ , \tag{E.4}$$

with an n-by-n unitary matrix u. ■
Therefore we find,

$$\mathrm{Inv}_+(n) \simeq U(n)/O(n)\ . \tag{E.5}$$

Proof 1-s: It is easy to show that an arbitrary symmetric matrix can be rewritten as

$$A \to A' = u'Au'^{\mathrm{T}} = \begin{pmatrix} |a_1| & b_1 & 0 & 0 & \cdots \\ b_1 & |a_2| & b_2 & 0 & \cdots \\ 0 & b_2 & \ddots & \ddots & \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \end{pmatrix} \in \mathrm{Inv}_+(n)\ , \tag{E.6}$$

with an unitary matrix u'. The matrix A' is also a unitary matrix and this fact leads to $b_1 = 0$ or $b_2 = 0$. Therefore

$$A' = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A_{(n-1)} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} A_{(2)} & \mathbf{0} \\ \mathbf{0} & A_{(n-2)} \end{pmatrix}, \tag{E.7}$$

where $A_{(m)} \in \mathrm{Inv}_{+}(m)$. Recursively, we find A' takes a block-diagonal form whose diagonal elements are 1 or 2-by-2 symmetric unitary matrices. By using Proposition (E.2), we can show that there exists a unitary matrix $\tilde{u}$ such that $\tilde{u}A\tilde{u}^{\mathrm{T}} = \mathbf{1}_n$, that is, there exists a unitary matrix u such that $A = u^{\mathrm{T}}u$. ∎

By using a similar algorithm, we can show that

Theorem 1-a: An arbitrary $A \in \mathrm{Inv}_{-}(2m)$ can be rewritten as

$$A = u^{\mathrm{T}} J_m^- u, \quad J_m^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{1}_m, \tag{E.8}$$

with an appropriate unitary matrix, u, ($uu^\dagger = \mathbf{1}_{2m}$). ∎

Therefore we find

$$\mathrm{Inv}_{-}(2m) \simeq U(2m)/USp(2m). \tag{E.9}$$

A choice of $J_\epsilon \in \mathrm{Inv}_\epsilon(n)$ defines a subgroup $G_\epsilon(J_\epsilon)$ of $U(n)$ as

$$G_\epsilon(J_\epsilon) = \left\{ g \in U(n) \,\middle|\, g^{\mathrm{T}} J g = J \right\}. \tag{E.10}$$

Conversely, we can say that J_ϵ is an invariant tensor of $G_\epsilon(J_\epsilon)$.

Corollary 1: Two arbitrary elements $J, J' \in \mathrm{Inv}_\epsilon(n)$ are related to each other with an appropriate unitary matrix u as, $J' = u\,J\,u^{\mathrm{T}}$ thus the corresponding groups $G_\epsilon(J)$ and $G_\epsilon(J')$ are isomorphic to each other. ∎

Therefore, from Eq. (E.4) and Eq. (E.8) we find that $G_+(J_+)$ is isomorphic to $O(n)$ and $G_-(J_-)$ is isomorphic to $USp(n = 2m)$.

E.1 Diagonalization of the vacuum configuration

Theorem 2-s: Let us consider an arbitrary n-by-m $(n \le m)$ matrix Q satisfying

$$QQ^\dagger = (QQ^\dagger)^{\mathrm{T}}. \tag{E.11}$$

Then Q can always be decomposed as

$$Q = O \left(\begin{array}{ccc|ccc} \lambda_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \lambda_n & 0 & \cdots & 0 \end{array} \right) U, \tag{E.12}$$

where $O \in SO(n)$ with $J = \mathbf{1}_n$ and $U \in U(m)$.

Proof 2-s: Since $QQ^\dagger$ is symmetric and Hermitian, $QQ^\dagger$ is a real symmetric matrix. Therefore it can be diagonalized as $QQ^\dagger = O\Lambda^2 O^{\mathrm{T}}$ with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_\mathrm{n})$ with $\lambda_i \in \mathbb{R}_{\ge 0}$. ∎

Theorem 2-a: Let us consider an arbitrary $2n$-by-m $(2n \leq m)$ matrix Q satisfying

$$JQQ^\dagger = (QQ^\dagger)^{\mathrm{T}} J \,, \tag{E.13}$$

with $J = i\sigma_2 \otimes \mathbf{1}_n$. Then Q can always be decomposed as

$$Q = O \left(\begin{array}{c|ccc} & 0 & \cdots & 0 \\ \Lambda & \vdots & \ddots & \vdots \\ & 0 & \cdots & 0 \end{array} \right) U \,, \tag{E.14}$$

where $O \in USp(2n)$ and $U \in U(m)$ and $\Lambda = \mathbf{1}_2 \otimes \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_\mathrm{n})$ with $\lambda_i \in \mathbb{R}_{\geq 0}$. ■
Proof 2-a: The Hermitian positive semi-definite matrix $QQ^\dagger$ is always diagonalized as $QQ^\dagger = u\Lambda^2 u^\dagger$ with an appropriate unitary matrix $u \in U(2n)$. Then the condition tells us that $X = u^{\mathrm{T}} J u$ commutes with Λ^2, $[X, \Lambda^2] = 0$. We can set Λ to be positive semi-definite, then $[X, \Lambda] = 0$. Furthermore, we find $XX^\dagger = \mathbf{1}_{2n}$ and $X^{\mathrm{T}} = -X$. According to Theorem 1-a, thus, X turns out to be $X = u^{\mathrm{T}} J u = J$ by taking an appropriate u. This means u is an element of $USp(2n)$. Here Λ takes a form $\mathbf{1}_2 \otimes \Lambda'$, since $[\Lambda, J] = 0$. ■

E.2 Diagonalization of a non-Hermitian (anti)symmetric matrix

Theorem 3: An arbitrary n-by-n (anti)symmetric matrix M, (that is, $M^{\mathrm{T}} = \epsilon M$) can be written in a block-diagonal form as

$$M \;=\; u \begin{pmatrix} |\mu_{(1)}| J_{(1)} & & \\ & |\mu_{(2)}| J_{(2)} & \\ & & \ddots \end{pmatrix} u^{\mathrm{T}} \,, \tag{E.15}$$

where $J_{(k)} \in \mathrm{Inv}_\epsilon(n_k)$ and $n = \sum_k n_k$. ■
Proof 3: $MM^\dagger$ is an Hermitian matrix and thus, can always be diagonalized as

$$MM^\dagger = u \,\mathrm{diag}\left(|\mu_{(1)}|^2 \mathbf{1}_{n_1},\, |\mu_{(2)}|^2 \mathbf{1}_{n_2},\, \cdots \right) u^\dagger \,, \tag{E.16}$$

with a unitary matrix u and $|\mu_{(i)}| < |\mu_{(i+1)}|$. Therefore, $\tilde{M} \equiv u^\dagger M u^*$ satisfies

$$\begin{aligned} \tilde{M}\tilde{M}^\dagger &= \mathrm{diag}\left(|\mu_{(1)}|^2 \mathbf{1}_{n_1},\, |\mu_{(2)}|^2 \mathbf{1}_{n_2},\, \cdots \right) \\ &= (\tilde{M}\tilde{M}^\dagger)^{\mathrm{T}} = \tilde{M}^\dagger \tilde{M} \,. \end{aligned} \tag{E.17}$$

Note that $\tilde{M}^{\mathrm{T}} = \epsilon \tilde{M}$. This equation means that $\tilde{M}$ is a normal matrix $[\tilde{M}, \tilde{M}^\dagger] = 0$ and can be diagonalized as

$$\tilde{M} = \tilde{u} \,\mathrm{diag}\left(\mu_1,\, \mu_2,\, \cdots \right) \tilde{u}^\dagger \,, \tag{E.18}$$

with a unitary matrix $\tilde{u}$. By substituting this form into Eq.(E.17), we find that

$$|\mu_{(1)}|^2 = |\mu_1|^2 = |\mu_2|^2 = \cdots \,, \quad |\mu_{(2)}|^2 = |\mu_{n_1+1}|^2 = \cdots \,, \quad |\mu_{(3)}|^2 = \cdots \,. \tag{E.19}$$

and $\tilde{u}$ should take a block-diagonal form as

$$\tilde{u} = \mathrm{diag}(u_{(1)}, u_{(2)}, \cdots) \ , \tag{E.20}$$

where $u_{(k)}$ is an n_k-by-n_k unitary matrix. Therefore, $\tilde{M}$ also takes block-diagonal form as

$$\tilde{M} = \mathrm{diag}\left(|\mu_{(1)}|J_{(1)},\ |\mu_{(2)}|J_{(2)},\ \cdots\right) \ . \tag{E.21}$$

■

The meson field is always 'diagonalized' by fixing the flavor symmetry. Combining Theorem 1-s(1-a) with Theorem 3, we find the following corollaries.

Corollary 3-s: An arbitrary symmetric matrix M can be diagonalized

$$M = u\,m\,u^{\mathrm{T}}, \quad m = \mathrm{diag}(|\mu_1|,\ |\mu_2|,\ \cdots) \ , \tag{E.22}$$

with a unitary matrix u. ■

Corollary 3-a: An arbitrary anti-symmetric matrix M can be diagonalized

$$M = u\,m\,u^{\mathrm{T}}, \quad m = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathrm{diag}(|\mu_1|,\ |\mu_2|,\ \cdots) \ , \tag{E.23}$$

with a unitary matrix u. ■

Corollary 3': An arbitrary n-by-n (anti-)symmetric matrix M can be decomposed as

$$M = Q^{\mathrm{T}} J Q \ . \tag{E.24}$$

where Q is an n-by-m matrix and $J \in \mathrm{Inv}_\epsilon(m)$ with $m = \mathrm{rank}(M)$. ■

The (anti)symmetric matrix M breaks the $U(n)$ symmetry $M \to uMu^{\mathrm{T}}$ as

$$U(n) \quad \to \quad \left\{ \begin{array}{l} U(n_0) \times O(n_1) \times O(n_2) \times \cdots \\ U(n_0) \times USp(2m_1) \times USp(2m_2) \times \cdots \end{array} \right. , \tag{E.25}$$

where n_0 is the number of zero-eigenvalues of M.

APPENDIX $\mathfrak{F}$

Non-trivial uniqueness proof

In this section, we prove the uniqueness of the solution to Eq. (8.19). Here we consider the $SO(N)$ case. We can always write the N-by-N_{F} matrix Q as

$$Q = \left(\hat{Q},\, \mathbf{0}\right)\mathcal{U}, \quad \mathcal{U} \in U(N_{\mathrm{F}})\,, \tag{F.1}$$

up to $U(N)$ transformations which rotates the columns of the N-by-N matrix of $\hat{Q}$. We can show that for $\hat{M} \equiv \hat{Q}^{\mathrm{T}} J \hat{Q}$

$$\begin{aligned} \operatorname{rank}\hat{M} = N \qquad &\Leftrightarrow \qquad \operatorname{rank}\hat{Q} = N\,, \\ \operatorname{rank}\hat{M} = N-1 \qquad &\Rightarrow \qquad \operatorname{rank}\hat{Q} = N-1\,, \end{aligned} \tag{F.2}$$

since $\det \hat{M} = \det J (\det \hat{Q})^2$ and $N \geq \operatorname{rank}\hat{Q} \geq \operatorname{rank}\hat{M}$ is always satisfied.

F.1 Solution with $\operatorname{rank} M = N$

If the rank of $M \equiv Q^{\mathrm{T}} J Q$ is N, then $\hat{M}$ also has rank N. Therefore $\operatorname{rank}\hat{Q} = N$, namely $\hat{Q}$ is invertible and

$$U_Q \equiv \hat{Q}^{-1}\sqrt{\hat{Q}\hat{Q}^{\dagger}}\,, \tag{F.3}$$

is a unitary matrix, $U_Q \in U(N)$. In terms of this unitary matrix, we rewrite Eq.(8.19) as

$$\begin{aligned} X &= \sqrt{QQ^{\dagger}}e^{-V'}\sqrt{QQ^{\dagger}} = U_Q^{\dagger}\hat{Q}^{\dagger}e^{-V'}\hat{Q}U_Q\,, \\ X^2 &= \left(Q^{\mathrm{T}} J\sqrt{QQ^{\dagger}}\right)^{\dagger} Q^{\mathrm{T}} J \sqrt{QQ^{\dagger}} = U_Q^{\dagger}\hat{Q}^{\dagger}J^{\dagger}Q^{*}Q^{\mathrm{T}}J\hat{Q}U_Q = U_Q^{\dagger}\hat{M}^{\dagger}\hat{M}U_Q\,. \end{aligned} \tag{F.4}$$

Since $\hat{Q}$ and $\hat{M}$ are invertible, we find a unique solution of V'

$$V' = \log\left(\hat{Q}\frac{\mathbf{1}_N}{\sqrt{\hat{M}^{\dagger}\hat{M}}}\hat{Q}^{\dagger}\right)\,. \tag{F.5}$$

F.2 Solution with $\mathrm{rank}\, M = N - 1$

In this case $\mathrm{rank}\, \hat{Q} = N - 1$, we can use the $U(N)$ rotation so that the N-by-N matrix $\hat{Q}$ takes the form

$$\hat{Q} = \left(\begin{array}{c|c} & 0 \\ \tilde{Q} & \vdots \\ & 0 \end{array} \right) , \tag{F.6}$$

where $\tilde{Q}$ is an N-by-$(N-1)$ matrix. We can introduce an N-component vector p such that

$$p^{\mathrm{T}} J \hat{Q} = p^{\mathrm{T}} J Q = 0 \,, \quad p^{\mathrm{T}} J p = 1 \,, \tag{F.7}$$

and the following N-by-N matrix has the maximal rank

$$R \equiv \left(\tilde{Q}, p \right) \in GL(N, \mathbb{C}) \,. \tag{F.8}$$

Note that with a given $\tilde{Q}$, the column vector p is uniquely determined up to a sign. Since R is invertible, $e^{V'}$ can be decomposed as

$$e^{V'} = R \begin{pmatrix} B & c \\ c^{\dagger} & a \end{pmatrix} R^{\dagger} \,. \tag{F.9}$$

Here, B is an $(N-1)$-by-$(N-1)$ Hermitian matrix and a is a real parameter. Eq. (8.18) can be rewritten as

$$e^{V'^{\mathrm{T}}} J Q Q^{\dagger} = Q^{*} Q^{\mathrm{T}} J e^{V'} \,. \tag{F.10}$$

Substituting the above decomposition and multiplying by $R^{\mathrm{T}} J^{*}$ from the left and by $J^{\dagger} R^{*}$ from the right, we find that

$$B^{\mathrm{T}} \hat{M} = \hat{M} B, \quad c = 0 \,. \tag{F.11}$$

From the condition for $e^{V'} \in SO(N)^{\mathbb{C}}$, we find the following equations

$$a^2 = 1, \qquad \hat{M}^{\dagger} B^{\mathrm{T}} \hat{M} B = \mathbf{1}_{N-1} \,. \tag{F.12}$$

Note that we can say that B and a are positive definite since $c = 0$. Combining the above two equations, we obtain

$$B = \frac{\mathbf{1}_{N-1}}{\sqrt{\hat{M}^{\dagger} \hat{M}}} \,, \qquad a = 1 \,. \tag{F.13}$$

Therefore we finally find a unique solution

$$e^{V'} = \tilde{Q} \frac{\mathbf{1}_{N-1}}{\sqrt{\hat{M}^{\dagger} \hat{M}}} \tilde{Q}^{\dagger} + p p^{\dagger} \,. \tag{F.14}$$

Note that $pp^{\dagger}$ is uniquely determined for a given $\tilde{Q}$, namely for a given Q. Even if we could construct a similar solution for V' in the case of $\mathrm{rank}\, M < N - 1$, it is obviously expected that a matrix corresponding to $pp^{\dagger}$ would not be unique. These results exactly reflect the appearance of a partial Coulomb phase in the case of $\mathrm{rank}\, M < N - 1$.

APPENDIX $\mathfrak{G}$

Deformed Kähler potential for $USp(2M)$

The expansion of the deformed Kähler potential of Eq. (8.39) reads

$$
\begin{aligned}
&K_{USp,\mathrm{deformed}} = \\
&\quad \frac{1}{2}\sum_{i,j}\frac{1}{\mu'_i+\mu'_j}\left[1+\frac{\varepsilon^2}{\mu'_i\mu'_j}\right]\phi_{ij}\phi^\dagger_{ji} \\
&\quad -\frac{1}{2}\sum_{i,j,k}\frac{\mu_i}{(\mu'_i+\mu'_j)(\mu'_i+\mu'_k)(\mu'_j+\mu'_k)}\left[1+\varepsilon^2\frac{\mu'_i+\mu'_j+\mu'_k}{\mu'_i\mu'_j\mu'_k}\right]\phi_{ij}\phi^\dagger_{jk}(\phi J^\dagger)_{ki}+\text{c.c.} \\
&\quad +\sum_{i,j,k,l}X(\varepsilon)_{ijkl}(\phi J^\dagger)_{ij}(\phi J^\dagger)_{jk}\phi_{kl}\phi^\dagger_{li}+\text{c.c.} \\
&\quad +\frac{1}{2}\sum_{i,j,k,l}\frac{\mu_j\mu_l}{P'_{ijkl}}\left[C^{(1)'}_{ijkl}+\varepsilon^2\frac{C^{(1)'}_{ijkl}C^{(2)'}_{ijkl}-C^{(3)'}_{ijkl}}{C^{(4)'}_{ijkl}}\right](\phi J^\dagger)_{ij}\phi_{jk}\phi^\dagger_{kl}(J\phi^\dagger)_{li} \\
&\quad -\frac{1}{4}\sum_{i,j,k,l}\left[\frac{C^{(3)'}_{ijkl}}{P'_{ijkl}}+2\varepsilon^2\frac{C^{(1)'}_{ijkl}}{P'_{ijkl}}+\varepsilon^4\frac{C^{(1)'}_{ijkl}C^{(2)'}_{ijkl}-C^{(3)'}_{ijkl}}{C^{(4)'}_{ijkl}P'_{ijkl}}\right]\phi_{ij}\phi^\dagger_{jk}\phi_{kl}\phi^\dagger_{li} \\
&\quad +\text{Kähler trfs.}+\mathcal{O}(\phi^5)\,, \qquad\qquad \text{(G.1)}
\end{aligned}
$$

where $\mu_i'^2 \equiv \mu_i^2+\varepsilon^2$.

The resulting curvature is

$$
\begin{aligned}
R|_{\phi=0} = & -2\sum_{i}^{M} \frac{\mu_i'^6 + \varepsilon^2 7\mu_i'^4 - \varepsilon^4 17\mu_i'^2 - \varepsilon^6 7}{2\mu_i'(\varepsilon^2+\mu_i'^2)^3} \\
& -2\sum_{i,j}^{M} \frac{\mu_i'^4\mu_j'^4}{(\mu_i'+\mu_j')(\varepsilon^2+\mu_i'^2)(\varepsilon^2+\mu_j'^2)(\varepsilon^2+\mu_i'\mu_j')^2} \\
& +2\varepsilon^2\sum_{i,j}^{M} \frac{\mu_i'^2\mu_j'^2(\mu_i'^2+\mu_j'^2)}{(\mu_i'+\mu_j')(\varepsilon^2+\mu_i'^2)(\varepsilon^2+\mu_j'^2)(\varepsilon^2+\mu_i'\mu_j')^2} \\
& -4\varepsilon^4\sum_{i,j}^{M} \frac{\mu_i'^2\left(6\mu_i'^2+9\mu_i'\mu_j'+5\mu_j'^2\right)}{(\mu_i'+\mu_j')(\varepsilon^2+\mu_i'^2)(\varepsilon^2+\mu_j'^2)(\varepsilon^2+\mu_i'\mu_j')^2} \\
& -4\varepsilon^6\sum_{i,j}^{M} \frac{\mu_i'\left(5\mu_i'^2+15\mu_i'\mu_j'+13\mu_j'^2\right)}{\mu_j'(\mu_i'+\mu_j')(\varepsilon^2+\mu_i'^2)(\varepsilon^2+\mu_j'^2)(\varepsilon^2+\mu_i'\mu_j')^2} \\
& -2\varepsilon^8\sum_{i,j}^{M} \frac{10\mu_i'+13\mu_j'}{\mu_j'(\mu_i'+\mu_j')(\varepsilon^2+\mu_i'^2)(\varepsilon^2+\mu_j'^2)(\varepsilon^2+\mu_i'\mu_j')^2} \\
& +16\sum_{i,j,k}^{M} \frac{\mu_i'^3\mu_j'^2\mu_k'^3}{(\mu_i'+\mu_j')(\mu_i'+\mu_k')(\mu_j'+\mu_k')(\varepsilon^2+\mu_i'\mu_j')(\varepsilon^2+\mu_i'\mu_k')(\varepsilon^2+\mu_j'\mu_k')} \\
& +16\varepsilon^2\sum_{i,j,k}^{M} \frac{\mu_i'^2\mu_j'^2\mu_k'^2}{(\mu_i'+\mu_j')(\mu_i'+\mu_k')(\mu_j'+\mu_k')(\varepsilon^2+\mu_i'\mu_j')(\varepsilon^2+\mu_i'\mu_k')(\varepsilon^2+\mu_j'\mu_k')} \\
& +16\varepsilon^4\sum_{i,j,k}^{M} \frac{\mu_i'\mu_k'\left(4\mu_i'^2\mu_j'+2\mu_i'^2\mu_k'+6\mu_i'\mu_j'^2+3\mu_j'^3+4\mu_i'\mu_j'\mu_k'\right)}{\mu_j'(\mu_i'+\mu_j')(\mu_i'+\mu_k')(\mu_j'+\mu_k')(\varepsilon^2+\mu_i'\mu_j')(\varepsilon^2+\mu_i'\mu_k')(\varepsilon^2+\mu_j'\mu_k')} \\
& +16\varepsilon^6\sum_{i,j,k}^{M} \frac{\mu_i'^2(\mu_j'+\mu_k')+\mu_j'(\mu_j'+\mu_k')^2+\mu_i'(2\mu_j'^2+2\mu_j'\mu_k'+\mu_k'^2)}{\mu_j'(\mu_i'+\mu_j')(\mu_i'+\mu_k')(\mu_j'+\mu_k')(\varepsilon^2+\mu_i'\mu_j')(\varepsilon^2+\mu_i'\mu_k')(\varepsilon^2+\mu_j'\mu_k')}\ .
\end{aligned}
\tag{G.2}
$$

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